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SEMILOCALITY AND DENSITY OF MULTI-ANISOTROPIC SOBOLEV SPACES

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Abstract

In this paper, we prove the semilocality of some multi-anisotropic Sobolev spaces, the density of smooth finite functions in those spaces when Sobolev spaces generated by completely regular Newton polyhedrons and give some examples showing that multi-anisotropic Sobolev space generated by a non-regular Newton polyhedron is not semilocal.

1. Introduction

We use the following standard notation: N denotes the set of all natural numbers $N_0^n = N_0 \times \ldots \times N_0$ (where $N_0 = N \cup \{0\}$) is the set of all n-dimensional multi-indices, E^n and R^n are the n-dimensional Euclidean spaces of points (vectors) $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$,

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respectively, $R^{n,+} = \{\xi : \xi \in R^n, \xi_j \geq 0 (j = 1, \dots, n)\}$, $R^{n,0} = \{\xi : \xi \in R^n, \xi_1 \dots \xi_n \neq 0\}$.

For $\xi \in R^n$, $x \in E^n$ and $\alpha \in R^{n,+}$, we put $|\xi| = \sqrt{\xi_1^2 + \dots + \xi_n^2}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where $D_j = \frac{1}{i} \frac{\delta}{\delta x_j}$ $(j = 1, \dots, n)$, $\alpha \in N_0^n$.

Let $\mathcal{A} = \{\alpha^j = (\alpha_1^j, \dots, \alpha_n^j)\}_1^M$ be a finite set of points in $R^{n,+}$. By the *Newton polyhedron* of the set $\mathcal{A}$, we mean the minimal convex hull (which is a polyhedron) $\Re = \Re(\mathcal{A})$ in $R^{n,+}$ containing all points of $\mathcal{A}$.

A polyhedron $\Re$ with vertices in $R^{n,+}$ is said to be *complete* (see [17] or [9]), if $\Re$ has a vertex at the origin and one vertex (distinct from the origin) on each coordinate axis of $R^{n,+}$. The k-dimensional faces of a polyhedron $\Re$ are denoted by $\Re_i^k (i = 1, \dots, M_k', k = 0, 1, \dots, n-1)$. The set of 0-dimensional faces (vertices) of $\Re$ we denote by $\Re^0$.

In the sequel, the outward (with respect to $\Re$) normal to the hyper plane of the support of the complete polyhedron $\Re$ containing some face $\Re_i^k$ and not containing any other face of dimension greater than k will be called simply the *outward normal* to the face $\Re_i^k$. Thus, a given vector λ can serve as an outward normal to one and only one face of a convex complete polyhedron $\Re$.

The face $\Re_i^k (1 \leq i \leq M_k', 0 \leq k \leq n-1)$ of a polyhedron $\Re$ is said to be *principal* (see [17]) if there is an outward normal with at least one positive component among outward normals to the face. If moreover, there is an outward normal with nonnegative (positive) components, the face $\Re_i^k$ is said to be *regular* (*completely regular*). A complete polyhedron $\Re$ is said to be *regular* (*completely regular*), if all its non-coordinate

$(n-1)$-dimensional faces are regular (completely regular) (see [12], [2], or [5], [7]).

Let $\Re$ be a complete polyhedron with vertices in N_0^n, $\Re^0$ be the set of its vertices, Ω be a domain in E^n, and $1 < p < \infty$. Denote by $W_p^{\Re}(\Omega)$ (respectively $W_p^{\Re^0}(\Omega)$) the set of functions u with the following bounded norms (see [12] or [2], paragraph 13):

$$\|u\|_{W_p^{\Re}(\Omega)} = \sum_{\alpha \in \Re} \|D^\alpha u\|_{L_p(\Omega)}, \tag{1.1}$$

and respectively,

$$\|u\|_{W_p^{\Re^0}(\Omega)} = \sum_{\alpha \in \Re^0} \|D^\alpha u\|_{L_p(\Omega)}. \tag{1.2}$$

For a vector $m = (m_1, \dots, m_n) \in E^n$, $m_j > 0 (j = 1, \dots, n)$ the collections

$$A_1 = \{\alpha : \alpha \in N_0^n; \left|\frac{\alpha}{m}\right| \equiv \sum_{j=1}^n \frac{\alpha_j}{m_j} \leq 1\},$$

and

$$A_2 = \{(0, \dots, 0) \cup [\alpha : \alpha \in N_0^n; \left|\frac{\alpha}{m}\right| = 1]\},$$

the sets $\Re^0(A_1)$ and $\Re^0(A_2)$ coincide, where the sets $W_p^{\Re(A_1)}(\Omega)$ (respectively, $W_p^{\Re(A_2)}(\Omega)$) coincide with the isotropic Sobolev space $W_p^m(\Omega)$ when $m_1 = m_2 = \dots = m_n$ (respectively with the anisotropic Sobolev space $\widetilde{W}_p^m(\Omega)$ when $m_i \neq m_j$ for a pair (i, j)) with the norm

$$\|u\|_{W_p^m} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega)} (\|u\|_{\widetilde{W}_p^m} = \sum_{|\alpha| = m} \|D^\alpha u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}).$$

Therefore, the sets $W_p^{\Re}(\Omega)$ with the arbitrary polyhedron $\Re$ and with the suitable norms we will call multi-anisotropic Sobolev spaces.

The notion of completely regular polyhedron, not being right triangles, arises in connection with numerous problems in the theory of partial differential equations, in particular when we study hypoelliptic (see [11], Definition 11.1.2) or hyperbolic (see, for instance, [10] or [11], Definition 12.3.3) differential operators (equations). Recall that a linear differential operator $P(D) = P(D_1, \dots, D_n) = \sum_\alpha \gamma_\alpha D^\alpha$ with constant coefficients (here the sum goes over a finite set of multi-indices $(P) = \{\alpha \in N_0^n;\ \gamma_\alpha \neq 0\}$) is called *hypoelliptic* if its complete symbol (characteristic polynomial) $P(\xi) = P(\xi_1, \dots, \xi_n) = \sum_\alpha \gamma_\alpha \xi^\alpha$ satisfies the condition $|D^\alpha P(\xi)| / |P(\xi)| \to 0$ as $|\xi| \to \infty$ for all $0 \neq \alpha \in N_0^n$. Operator $P(D)$ is called N-hyperbolic (by Gȯrding) if there exists a real number τ_0 such that $P(\xi + \tau N) \neq 0$ for all $\xi \in R^n$ and $\tau < \tau_0$ (see, for instance, [10] or [11], Definition 12.3.3). It is well known that an operator $P(D)$ is hypoelliptic if and only if all distributional solutions of the equation $P(D)u = 0$ are infinitely differentiable (see [11], Theorem 11.1.3) and that Cauchy problem is well posed for a large set of hyperbolic by Gȯrding operators and s-hyperbolic operators (see, for instance, [16], [9], [21], [5], [7], [23] and others).

Operator $P(D)$ is called almost hypoelliptic (see [13]) if there exists a constant $C > 0$ such that $|D^\alpha P(\xi)| / [|P(\xi)| + 1] \leq C$ for all $\alpha \in N_0^n$ and $\xi \in R^n$.

The Newton polyhedron of the set $(P) \cup 0$ is called the *Newton* (or *characteristic*) *polyhedron* of the operator $P(D)$ (the polynomial $P(\xi)$) (see [13], [17], [9] or [19]).

It turned out that there is a strong connection between the (almost) hypoellipsity of operator $P(D)$ and its Newton polyhedron: the Newton polyhedron of a hypoelliptic operator is completely regular and an almost hypoelliptic operator is regular. On the other hand, in [8], the following statement was proved: let f and all its derivatives be square integrable on E^n with a certain exponential weight. Then all square integrable (with the same weight) solutions of the equation $P(D)u = f$ have square integrable derivatives with this weight if and only if the operator $P(D)$ is almost hypoelliptic. In other words, if f is infinitely differentiable, then all distributional solutions of the equation $P(D)u = f$ which belong to the certain weighted multi-anisotropic Sobolev space $W_{p,\delta}^{\Re}$ (the definition see below in Section 3) are infinitely differentiable if and only if $P(D)$ is almost hypoelliptic.

Newton polyhedron generalizes the notion of degree of polynomial of n variables and the notion of degree of partial differential equations. There are great many applications of Newton polyhedron's concept to different fields of mathematics (see, for instance, [14]-[20] and others) but in this work we will be concentrate only to (weighted) multi-anisotropic Sobolev spaces generated by some completely regular Newton polyhedron.

Sobolev spaces play an outstanding role in modern analysis. In particular, weighted Sobolev spaces are of great interest in many fields of mathematics and first of all they arise in various issues of the theory of partial differential equations. Many monographs and papers have already been devoted to this topic. We mention only some of such works which are closely related to the present paper. First of all, we refer to the monographs [2], [11], and [22]. In these monographs, it is proved semilocality of various (weighted) Banach spaces, in particular, classical Sobolev spaces.

In the paper [6] by Carlson and Maz'ya, necessary and sufficient conditions are given for a function from weighted Sobolev spaces (with a weight μ which specifies a non trivial positive Radon measure) to be approximated by test functions. Besov proved in [1], the density of the infinitely differentiable finite functions in some weighted Sobolev space. Burenkov proved in [3] (see also [4]), the density of finite functions in the isotropic Sobolev space $W_p^l(\Omega)$ for any open set Ω. In book [15], Kufner deals with properties of weighted Sobolev spaces $W_{p,\mu}^m(\Omega)$ the weight function μ being dependent on $d(x, \partial\Omega)$, the distance of points of the domain Ω from its boundary (or its part).

These works are devoted to isotropic (or anisotropic) weighted Sobolev spaces, i.e., the spaces which are generated by a homogeneous (or, respectively non homogeneous) vector $m = (m_1, \ldots, m_n)$. Its Newton polyhedrons are $(n+1)$-simplexes (geometrically, for example, in case $n = 2$, they are right triangles with a vertex in the origin, isosceles or not).

Here we consider general case when the Sobolev space is generated by a Newton polyhedron of any kind.

It turned out that for a set $\mathcal{A}$ (say, polyhedrons $\Re$) the nature of a multi-anisotropic Sobolev space can be essentially different from usual (isotropic or anisotropic) Sobolev spaces. Therefore, a natural problem arises to find conditions on a polyhedron $\Re$ corresponding to the set $\mathcal{A}$ and on a domain Ω under which

(1) the norms (1.1) and (1.2) are equivalent, i.e., the spaces $W_p^{\Re}(\Omega)$ and $W_p^{\Re^0}(\Omega)$ coincide;

(2) the set $W_p^{\Re}(\Omega)$ is a semilocal space. Recall that a functional Banach space $B(\Omega)$ is called semilocal if $u \in B(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ leades $\varphi u \in B(\Omega)$ (see, for instance, [10], Definition 10.1.18);

(3) the set of infinitely differentiable functions with compact supports in Ω is dense in multi-anisotropic Sobolev space $W_p^{\Re}(\Omega)$.

It turned out that there is a direct connection between the geometric properties of a Newton polyhedron $\Re$ and the answers to these questions. We note in this regard that for Sobolev spaces $W_p^m(\Omega)$ with different domains Ω there are definite answers to the listed questions (see, for instance, [2], [4] or [15]). In particular, the spaces $W_p^m(\Omega)$ and $\widetilde{W}_p^m(\Omega)$ are isometrically isomorphic.

In this paper, we prove the following in main result

Theorem. *Let* $\Re$ *be a completely regular Newton polyhedron*, g *be an exponential weight function*, $\Omega \subset E^n$ *be a domain satisfying the rectangle condition*, *and* $p \in (1, \infty)$. *Then* (a) *the space* $W_p^{\Re}(\Omega)$ *is semilocal*, (b) *the set* C_0^{∞} *is dense in* $W_p^{\Re}$ *and* $W_{p, g_\delta}^{\Re}$.

We present some examples when spaces $W_p^{\Re}$ are not semilocal if they are generated by non-regular Newton polyhedrons.

2. Equivalence of Norms and Semilocality of Multi-Anisotropic Sobolev Spaces

We mention some statements on which we shall rely in the sequel.

Theorem I (Il'in) (see [12] or [2], Theorem 13.3.2')**.** *Let* $1 < p < \infty$, *the domain* Ω *satisfy the rectangle condition* (*see* [12] *or* [2], *p*. 13.1) *and let the Newton polyhedron* $\Re(\mathcal{A})$ *of a collection of multi-indices* $\mathcal{A} = \{e^1, \dots, e^{N_0}\}$ *be completely regular. Then there exists a constant* $C > 0$ *such that*

$$\|u\|_{W_p^{\Re}(\Omega)} \equiv \sum_{\nu \in \Re(\mathcal{A})} \|D^{\nu} u\|_{L_p(\Omega)} \leq C \sum_{i=1}^{N_0} \|D^{e^i} u\|_{L_p(\Omega)},$$

for all $u \in W_p^{\Re}(\Omega)$.

Since the inverse inequality is obvious, this implies that for a completely regular polyhedron and a domain Ω satisfying the rectangle condition, the norms (1.1) and (1.2) are equivalent, i.e., the spaces $W_p^{\Re}(\Omega)$ and $W_p^{\Re^0}(\Omega)$ coincide.

Definition 2.1 (see [2], p. 11)**.** A measurable function Φ is called L_p-multiplier (denoted by $\Phi \in M_p^p$), if the transformation $T_\Phi : L_p \to L_p$ defined by equality

$$T_\Phi f = \frac{1}{(2\pi)^{n/2}} \int_{E^n} \Phi(\xi) F[f](\xi) e^{i(x,\xi)} d\xi \equiv F^{-1}[\Phi F[f]]$$

is bounded for all functions $f \in C_0^\infty$, i.e., there exists a constant $C > 0$ such that $\|T_\Phi f\|_p \le C\|f\|_p$ for all $f \in C_0^\infty$.

Theorem L (Lizorkin, see [2], p. 11)**.** *A function* $\Phi \in C^n(R^{n,0})$ *is a* L_p*-multiplier* ($\Phi \in M_p^p$) *if there exists a number* $M > 0$ *such that* $|\xi_1^{k_1} \dots \xi_n^{k_n} D^k \phi(\xi)| \le M$ *for all* $\xi \in R^{n,0}$, *where* $k = (k_1, \dots, k_n)$ *and* k_j *takes only values* 0 *and* 1, $j = 1, \dots, n$.

Theorem M (Mikhailov, see [17])**.** *For any set* $\mathcal{A}$ *of points* $e^1, \dots, e^{N_0} \in R^{n,0}$ *with the Newton polyhedron* $\Re = \Re(\mathcal{A})$, *there exists a constant* $C = C(\Re) > 0$ *such that*

$$\sum_{\alpha \in \Re} |\xi^\alpha| \le C \sum_{i=1}^{N_0} |\xi^{e^i}|, \quad \forall \xi \in R^n.$$

Theorem 2.1. *For any completely regular polyhedron* $\Re$, *any domain* $\Omega \subset E^n$ *satisfying the rectangle condition and any* $p \in (1, \infty)$ *a space* $W_p^{\Re}(\Omega)$ *is semilocal.*

Proof. Let $u \in W_p^{\Re}(\Omega)$, $\varphi \in C_0^{\infty}(\Omega)$ are fixed and $\alpha \in \Re$ is arbitrary. By the Leibnitz formula, we conclude that

$$D^{\alpha}(u\varphi) = \sum_{\beta \leq \alpha} D^{\alpha - \beta} u D^{\beta} \varphi. \tag{2.1}$$

Since the polyhedron $\Re$ is completely regular, $\alpha - \beta \in \Re$ for any $\alpha \in \Re$ and $\beta \leq \alpha$. Therefore by Theorem I, there exists a constant $C_1 > 0$ such that for all $\alpha \in \Re$ and $\beta \leq \alpha$

$$\| D^{\alpha - \beta} u \|_{L_p(\Omega)} \leq C_1 \| u \|_{W_p^{\Re}(\Omega)}. \tag{2.2}$$

On the other hand, since $\varphi \in C_0^{\infty}(\Omega)$, there exists a constant $C_2 > 0$ such that for all $\alpha \in \Re$ and $\beta \leq \alpha$

$$| D^{\beta} \varphi(x) | \leq C_2, \quad \forall x \in \Omega. \tag{2.3}$$

It follows from relations (2.1)-(2.3) that there exists a constant $C_3 > 0$ such that

$$\| u\varphi \|_{W_p^{\Re}(\Omega)} \leq C_3 \| u \|_{W_p^{\Re}(\Omega)},$$

i.e., $u\varphi \in W_p^{\Re}(\Omega)$, which proves the theorem. □

Lemma 2.1. *Let $1 < p < \infty$ and $\Re = \Re(\mathcal{A})$ be the Newton polyhedron of a collection of multi-indices $\mathcal{A} = \{e^1, \ldots, e^{N_0}\}$. Then there exists a constant $C > 0$ such that for all $u \in C_0^{\infty}$*

$$\sum_{\nu \in \Re} \| D^{\nu} u \|_{L_p} \leq C \sum_{i=1}^{N_0} \| D^{e^i} u \|_{L_p}. \tag{2.4}$$

Proof. Perform the Fourier transformation to functions u from C_0^{∞}. Applying Theorem M and Parseval's equality we obtain inequality (2.4) for $p = 2$. To prove the inequality (2.4) for $p \neq 2$ note that by the well-known properties of Fourier transformation we have

$$F[D^{\nu}u] = \xi^{\nu}F[u];\; F[D^{e^j}u] = \xi^{e^j}F[u] \;\; (j = 1, \ldots, N_0).$$

A simple computation gives

$$F[D^{\nu}u] = \sum_{j=1}^{N_0} \phi_j(\xi)F[D^{e^j}u],$$

where

$$\Phi_j(\xi) = \frac{\xi^{\nu+e^j}}{\sum_{k=1}^{N_0} \xi^{2e^j}} \equiv \frac{\xi^{\nu+e^j}}{Q(\xi)} \quad (j = 1, \ldots, N_0).$$

To prove the inequality (2.4) for any $p \in (1, \infty)$ it is sufficient (by the definition of L_p-multipliers) to show that $\Phi_j \in M_p^p$ $(j = 1, \ldots, N_0)$. For this purpose we apply Theorem L.

The boundedness of $\{\Phi_j\}$ leads immediately from Theorem M. Let us show the boundedness of, for example, $\{|\xi_1 \frac{\partial \phi_j}{\partial \xi_1}|\}$.

Again, a simple computation gives for each $j = 1, \ldots, N_0$

$$\xi_1 \frac{\partial \phi_j}{\partial \xi_1} = \phi_j(\xi)[(e_1^j + \nu_1) - 2\sum_{k=1}^{N_0} e_1^k \frac{\xi^{2e^k}}{Q(\xi)}].$$

Since $|\xi^{2e^k}/Q(\xi)| \leq 1$ $(k = 1, \ldots, N_0)$ for all $\xi \in R^n$, this implies the boundedness of $\{|\xi_1 \frac{\partial \phi_j}{\partial \xi_1}|\}$. By the same way one can prove the boundedness of other derivatives. Lemma 2.1 is proved. □

Lemma 2.2. *Let the Newton polyhedron* $\Re = \Re(\mathcal{A})$ *of a collection* $\mathcal{A} = \{e^1, \dots, e^{N_0}\}$ *of multi-indices be completely regular. The set of infinitely differentiable finite* (*in* E^n) *functions is dense in* $W_p^{\Re} = W_p^{\Re}(E^n)$ *if and only if the inequality* (2.4) *is valid for all functions* $u \in W_p^{\Re}$.

Proof. Sufficiency. Let the inequality (2.4) is valid, $\omega \in C_0^{\infty}$ be a function of one variable such that $\omega(t) = 0$, outside of $(0, 1)$, and $\int_0^1 \omega(t)dt = 1$. Fix a function $u \in W_p^{\Re}$ and put

$$u_h(x) = \frac{1}{h^n} \int \prod_{i=1}^{n} \omega\left(\frac{y_i}{h} \right) u(x + y) dy.$$

Then it is easy to verify that (see, for example, [2], p. 5): (1) $u_h \in C^{\infty}$, (2) $\|u - u_h\|_{W_p^{\Re}} \to 0$ as $h \to 0$. Thus, the set of infinitely differentiable functions is dense in $W_p^{\Re}$, and it remains to proof that every infinitely differentiable function $u \in W_p^{\Re}$ can be approximated in $W_p^{\Re}$ by C_0^{∞} - functions.

Let $\chi_k \in C_0^{\infty}$ for any $k \in N$, and $0 \le \chi_k(x) \le 1$ for all $x \in E^n$, $\chi_k(x) = 1$ for $|x| \le k$, $\chi_k(x) = 0$ for $|x| > k + 1$, $|D^{\alpha}\chi_k(x)| \le M$, where the constant $M > 0$ does not depend on $\alpha \in N_0^n$ and k.

For any $k \in N$ and $u \in W_p^{\Re} \cap C^{\infty}$, let $\varphi_k = \chi_k u$. It is clear that $\varphi_k \in C_0^{\infty}$. On the other hand, it follows from Theorem 2.1 (recall that the polyhedron $\Re$ is completely regular, and obviously the space E^n satisfies the rectangle condition) that $\varphi_k \in W_p^{\Re}$. Then applying Leibnitz' formula we get for a number $C > 0$

$$\sum_{j=1}^{N_0} \|D^{e^j} u - D^{e^j} \varphi_k\|_{L_p} = \sum_{j=1}^{N_0} \|D^{e^j} [u - \varphi_k]\|_{L_p} = \sum_{j=1}^{N_0} D^{e^j} [u(1 - \chi_k)]\|_{L_p} (|x| > k)$$

$$\leq \sup_{x, k, \alpha \in \Re} |D^{\alpha} [1 - \chi_k(x)] \sum_{\nu \in \Re} \|D^{\nu} u\|_{L_p} (|x| > k) \leq C \sum_{\nu \in \Re} \|D^{\nu} u\|_{L_p} (|x| > k).$$

Since $u \in W_p^{\Re}$ hence $D^{\nu} u \in L_p$, $\nu \in \Re$. Consequently, $\|D^{\nu} u\|_{L_p(|x|>k)} \to 0$ as $k \to \infty$, i.e., $\|u - \varphi_k\|_{W_p^{\Re}} = \sum_{j=1}^{N_0} \|D^{e^j} u - D^{e^j} \varphi_k\|_{L_p} \to 0$ as $k \to \infty$. The sufficiency is proved.

Necessity. The inequality (2.4) is valid for functions from C_0^{∞} by Lemma 2.1. Since the set C_0^{∞} is dense in $W_p^{\Re}$, then the inequality (2.4) is valid for all functions $u \in W_p^{\Re}$. So Lemma 2.2 is proved. □

Combining Lemmas 2.1 and 2.2, we obtain

Theorem 2.2. *Let the Newton polyhedron* $\Re$ *of a set of multi-indices* $e^1, \ldots, e^{N_0}$ *be completely regular. Then the set* C_0^{∞} *is dense in* $W_p^{\Re}$.

We present two examples showing that the multi-anisotropic Sobolev space $W_p^{\Re}(\Omega)$ corresponding to a non regular Newton polyhedron $\Re$ may be non semilocal. The first example related to a bounded and the second to an unbounded domain.

Example 1. Let $n = 2$ and $\Re$ be the Newton polyhedron of multi-indices $(0, 0), (1, 0), (0, 1), (2, 1) \in N_0^2$. It is easy to see that the quadrangle $\Re$ is non regular (in the sense of Introduction) and the projection (2, 0) of the vertex (2, 1) on the axis $0\alpha_1$ does not belong to $\Re$.

Let $u(x) = u(x_1, x_2) = x_1^{4/3} + x_2$, and $\Delta_1 = \{-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$. Then a simple computation shows that u, $D^{(1,0)}u$, $D^{(0,1)}u$, $D^{(2,1)}u$ belong to $L_2(\Delta_1)$, and $D^{(2,0)}u = \frac{4}{9} x_1^{-2/3} \notin L_2(\Delta_1)$.

Let $\psi \in C_0^{\infty}(\Delta_1)$, $\psi(x) = \psi(x_1, x_2) = x_2$ for $x \in \Delta_{1/2}$. Since $D^{(0,1)}\psi(x) = 1$ for $x \in \Delta_{1/2}$, it follows that $D^{(2,1)}(\psi(x)u(x)) = \frac{4}{9} D^{(0,1)}\psi(x)x_1^{-2/3} = \frac{4}{9}$ $x_1^{-2/3} \notin L_2(\Delta_1)$, i.e., $\psi u \notin W_2^{\Re}(\Delta_1)$, which meanes that $W_2^{\Re}(\Delta_1)$ is not semilocal.

Example 2. Let $\Re$ be as in Example 1, the function $f \in C_0^{\infty}(-1, 1)$ being chosen such that

$$A(f) \equiv \int_{-1}^{1} [f(t) + 5tf'(t) + 2t^2 f''(t)]^2 dt \neq 0. \tag{2.5}$$

Let also $u(x, y) = x_1^2 f(x_1^2 x_2)$, $\Omega = \{(x_1, x_2) \in E^2, |x_1| < 1, -\infty < x_2 < \infty\}$. Then

$$\|u\|_{L_2(\Omega)}^2 = \iint_{\Omega} x_1^4 f^2(x_1^2 x_2) dx_1 dx_2 = \int_{|x_1|<1} x_1^2 [\int_{-\infty}^{\infty} f^2(x_1^2 x_2) d(x_1^2 x_2)] dx_1$$

$$= \int_{|x_1|<1} x_1^2 [\int_{|x_1^2 x_2|<1} f^2(x_1^2 x_2) dx_2] dx_1 = \int_{-1}^{1} x_1^2 [\int_{-1}^{1} f^2(r) dr] dx_1 < \infty.$$

For $D^{1,0}u$, we have $D^{1,0}u = 2x_1 f(x_1^2 x_2) + x_1^2(2x_1 x_2) f'(x_1^2 x_2)$, where

$$\iint_{\Omega} |x_1 f(x_1^2 x_2)|^2 dx_1 dx_2 = \int_{|x_1|<1} [\int_{|x_1^2 x_2|<1} |f(x_1^2 x_2)|^2 d(x_1^2 x_2)] dx_1 < \infty,$$

$$\iint_{\Omega} x_1^2 |x_1^2 x_2|^2 |f'(x_1^2 x_2)|^2 dx_1 dx_2 = \int_{|x_1|<1} [\int_{r<1} r^2 |f'(r)|^2 dr] dx_1 < \infty.$$

Thus, $D^{(1,0)}u \in L_2(\Omega)$. Obviously, $D^{(0,1)}u(x_1, x_2) = x_1^4 f'(x_1^2 x_2)$, and so $D^{(0,1)}u \in L_2(\Omega)$. For $D^{(2,0)}u$ and $D^{(2,1)}u$, we have respectively, $D^{(2,0)}u(x_1, x_2) = 2f(x_1^2 x_2) + 10(x_1^2 x_2) f'(x_1^2 x_2) + 4(x_1^2 x_2)^2 f''(x_1^2 x_2)$,

$$D^{(2,1)}u(x_1, x_2) = x_1^2[12f'(x_1^2x_2) + 18(x_1^2x_2)f''(x_1^2x_1) + 4(x_1^2x_2)^2 f'''(x_1^2x_2)].$$

Denoting by $x_1 = x_1,\ x_1^2x_2 = r,\ h_k(r) = r^{k-1}f^{(k-1)}(r)\,(k = 1, 2, 3)$ we have for each $k = 1, 2, 3,$

$$\int\int_\Omega x_1^2|x_1^2x_2|^{2(k-1)}|h_k(x_1^2x_2)|^2 dx_1dx_2 = \int_{-1}^{1} x_1^2[\int_{-1}^{1}|h_k(r)|^2 dr]dx_1 < \infty,$$

i.e., $D^{(2,1)}u \in L_2(\Omega)$. As by the condition (2.5)

$$\int\int_\Omega |D^{(2,0)}u|^2 dx_1dx_2 = A(f)\int_{-1}^{1}\frac{dx_1}{x_1^2} = \infty,$$

then $D^{(2,0)}u \notin L_2(\Omega)$.

Taking a function $\psi \in C_0^\infty(\Omega)$, as in Example 1, where $\Delta_{1/2} \subset \Delta_1 \subset \Omega$, we get $D^{(2,1)}(\psi u) = \psi'_{x_2}D^{(2,0)}u + \psi D^{(2,1)}u = D^{(2,0)}u + \psi$ $D^{(2,1)}u$ for $x \in \Delta_{1/2}$. One can see as above that $\psi D^{(2,1)}u \in L_2(\Omega)$ and since $D^{(2,0)}u \notin L_2(\Omega)$ hence $D^{(2,1)}(\psi u) \notin L_2(\Omega)$, i.e., $W_2^{\Re}(\Omega)$ is not semilocal.

3. Weighted Multi-Anisotropic Sobolev Spaces

In this section, we consider a weighted multi-anisotropic Sobolev space $W_{p,\delta}^{\Re} = W_{p,g_\delta}^{\Re}(E^n)$ with a weight function g, which is defined as follows:

Let $\alpha \in N_0^n$ be an arbitrary multi-index and $g \in C^\infty = C^\infty(E^n)$ be any positive function such that (a) for some positive constants κ and κ_α

$$\kappa^{-1}e^{-|x|} \le g(x) \le \kappa e^{-|x|};\ |D^\alpha g_\delta(x)| \le \kappa_\alpha \delta^{|\alpha|}g_\delta(x)\ \forall x \in R^n, \tag{3.1}$$

where $g_\delta(x) = g(\delta x)$ for any $\delta > 0$.

(b) Let $T > 0$ and $S_T = \{x \in R^n : |x| < T\}$, then there exist positive numbers σ_1 and σ_2 such that for any $\delta > 0$ and $x \in R^n$

$$\sup_{y \in G} g_\delta(x + y) \leq \sigma_1 g_\delta(x), \ \sup_{y \in G} |g_\delta(x + y) - g_\delta(x)| \leq \sigma_2 T g_\delta(x). \tag{3.2}$$

Note that the regularization (averaging) of function

$$H(x) = \begin{cases} e^{-|x|} & \text{if} \quad |x| > 1, \\ e^{-1} & \text{if} \quad |x| \leq 1. \end{cases}$$

(see, for instance, [4], Section 5) can be taken as a function g.

Let $1 < p < \infty$ and $\delta > 0$. Denote by $L_{p,\delta} =: L_{p, g_\delta}(E^n)$ the set of locally integrable functions in E^n with a bounded norms

$$\|u\|_{L_{p,\delta}} =: \|ug_\delta\|_{L_2} = [\int_{E^n} |u(x)|^2 g_\delta^2(x) dx]^{\frac{1}{2}}, \tag{3.3}$$

and for any completely regular polyhedron $\Re$ with vertices in N_0^n denote by $W_{p,\delta}^{\Re}$ the set of functions $u \in L_{p,\delta}$ with a bounded norms

$$\|u\|_{W_{p,\delta}^{\Re}} =: \sum_{\alpha \in \Re} \|(D^\alpha u) g_\delta \| L_p = \sum_{\alpha \in \Re} \|D^\alpha u \| L_{p,\delta}. \tag{3.4}$$

Similarly to the result concerning non-weighted Sobolev spaces (see [2], Subsection 9.4 or [12], Theorem 2), the following result can be proved.

Lemma 3.1. *Let* $\Re$ *be any completely regular polyhedron and* $\nu \in N_0^n$ *be any interior point of* $\Re$. *Then for any* $\varepsilon > 0$ *there exists a number* $C(\varepsilon) > 0$ *such that*

$$\|D^\nu u \| L_{p,\delta} \leq \varepsilon \|u\|_{W_{p,\delta}^{\Re}} + C(\varepsilon) \|u\|_{L_{p,\delta}}, \quad \forall u \in W_{p,\delta}^{\Re} \tag{3.5}$$

(*definitions of the corresponding norms see in* (3.3) *and* (3.4)).

Lemma 3.2. *Let* $\Re$ *be any completely regular polyhedron. Then in* $W^{\Re}_{p,\delta}$, *one can introduce a norm*

$$\|u\|'_{W^{\Re}_{p,\delta}} = \sum_{\alpha\in\Re} \|D^{\alpha}(ug_{\delta})\|_{L_p},$$

which is equivalent to the norm (3.4).

Proof. By the Leibnitz' formula,

$$\sum_{\alpha\in\Re} D^{\alpha}(ug_{\delta}) = \sum_{\alpha\in\Re}[D^{\alpha}u]g_{\delta} + \sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta}D^{\alpha-\beta}uD^{\beta}g_{\delta}. \tag{3.6}$$

From (3.6), applying properties (3.1)-(3.2) of function g_{δ} and Lemma 3.2, we obtain

$$\|u\|'_{W^{\Re}_{p,\delta}} \le C_1\|u\|_{W^{\Re}_{p,\delta}}, \quad \forall u \in W^{\Re}_{p,\delta}, \tag{3.7}$$

with a positive constant $C_1 = C_1(\delta)$. To prove the inverse inequality, we can rewrite the formula (3.6) in the form

$$\sum_{\alpha\in\Re}[D^{\alpha}u]g_{\delta} = \sum_{\alpha\in\Re} D^{\alpha}(ug_{\delta}) - \sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta}D^{\alpha-\beta}uD^{\beta}g_{\delta}. \tag{3.6'}$$

Since $|\beta| > 0$ and the polyhedron $\Re$ is completely regular, all multi-indices $\alpha - \beta$ in the right-hand side of (3.6') are interior points of $\Re$. Then for any $\varepsilon > 0$ we can use inequality (3.5) for the second sum in the right-hand side of (3.6'), i.e., there exist some positive constants C_2 and C_3 independing on ε such that for all $u \in W^{\Re}_{p,\delta}$

$$\Big\|\sum_{\alpha\in\Re}\sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta}D^{\alpha-\beta}uD^{\beta}g_{\delta}\Big\|_{L_{p,\delta}} \le \varepsilon C_2\|u\|_{W^{\Re}_{p,\delta}} + C(\varepsilon)C_3\|u\|_{L_{p,\delta}}.$$

This (together with (3.6')) implies

$$\|u\|_{W^{\Re}_{p,\delta}} \leq \|u\|'_{W^{\Re}_{p,\delta}} + \varepsilon C_2 \|u\|_{W^{\Re}_{p,\delta}} + C(\varepsilon) C_3 \|u\|_{L_{p,\delta}}. \tag{3.8}$$

Choose the number $\varepsilon > 0$ such that $1 - \varepsilon C_2 > 0$, carry over the second term in the right-hand side of (3.8) to the left-hand side and divide both parts of received inequality by $1 - \varepsilon C_2 > 0$. Then we get

$$\|u\|_{W^{\Re}_{p,\delta}} \leq C_4 \|u\|'_{W^{\Re}_{p,\delta}} + C_5 \|u\|_{L_{p,\delta}}, \quad \forall u \in W^{\Re}_{p,\delta},$$

with some positive constants C_4 and C_5. This (together with inequality (3.7)) completes the proof.

Theorem 3.1. *Let* $\Re$ *be any completely regular polyhedron. The set* $C_0^\infty = C_0^\infty(E^n)$ *is dense in* $W^{\Re}_{p,\delta}$.

Proof. Let $u \in W^{\Re}_{p,\delta}$, $S_1 = \{x \in E^n : |x| < 1\}$, $\varphi \in C_0^\infty(S_1)$, $\varphi(x) \geq 0$, $\int \varphi(x) dx = 1$, $\varepsilon > 0$ and $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x / \varepsilon)$. Now we put

$$u_\varepsilon(x) = u * \varphi_\varepsilon = \int u(x - y) \varphi_\varepsilon(y) dy = \varepsilon^{-n} \int u(x - y) \varphi(y / \varepsilon) dy.$$

It is well known (see, for instance, [2], p. 5) that $u_\varepsilon \in C_0^\infty$ and $\|u - u_\varepsilon\|_{L_p} \to 0$ as $\varepsilon \to 0$. To complete the proof of the theorem we shall show that

$$\|u - u_\varepsilon\|_{W^{\Re}_{p,\delta}} \to 0 \text{ as } \varepsilon \to 0. \tag{3.9}$$

Since $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$, we have

$$\|u - u_\varepsilon\|_{W^{\Re}_{p,\delta}} = \sum_{\alpha \in \Re} \|D^\alpha(u - u_\varepsilon)\|_{L_{p,\delta}} = \sum_{\alpha \in \Re} \|[D^\alpha u - (D^\alpha u)_\varepsilon] g_\delta\|_{L_p}$$

$$\leq \sum_{\alpha \in \Re} \|(D^\alpha u) g_\delta - ((D^\alpha u) g_\delta)_\varepsilon\|_{L_p} + \sum_{\alpha \in \Re} \|((D^\alpha u) g_\delta)_\varepsilon - (D^\alpha u)_\varepsilon g_\delta\|_{L_p}. \tag{3.10}$$

Since $(D^{\alpha}u)g_{\delta} \in L_p$ for $u \in L_p$ and $\alpha \in \Re$, and a function in L_p is mean continuous (see, for instance, [2]), we get

$$\sum_{\alpha \in \Re} \|(D^{\alpha}u)g_{\delta} - ((D^{\alpha}u)g_{\delta})_{\varepsilon}\|_{L_p} \to 0 \text{ as } \varepsilon \to +0. \tag{3.11}$$

According to the inequality (3.10), the proof will be completed by showing that

$$A_{\varepsilon} \equiv \sum_{\alpha \in \Re} \|((D^{\alpha}u)g_{\delta})_{\varepsilon} - (D^{\alpha}u)_{\varepsilon} g_{\delta}\|_{L_p} \to 0 \text{ as } \varepsilon \to +0. \tag{3.12}$$

Since $\varphi_{\varepsilon} \in C_0^{\infty}(S_{\varepsilon})$ for any $\varepsilon > 0$ hence

$$A_{\varepsilon} = \sum_{\alpha \in \Re} \left\| \int (D^{\alpha}u)(x-y)[g_{\delta}(x-y) - g_{\delta}(x)]\varphi_{\varepsilon}(y)dy \right\|_{L_p}.$$

In view of the inequality (3.2), taking $T = \varepsilon$, it follows that

$$A_{\varepsilon} \leq \sigma_2 \varepsilon \sum_{\alpha \in \Re} \left\| \int (D^{\alpha}u)(x-y) g_{\delta}(x-y)\varphi_{\varepsilon}(y)dy \right\|_{L_p}.$$

Applying here Young's inequality, we get

$$A_{\varepsilon} \leq \sigma_2 \varepsilon \sum_{\alpha \in \Re} \|(D^{\alpha}u)g_{\delta}\|_{L_p} \|\varphi_{\varepsilon}\|_{L_1}.$$

Since $u \in W_{p,\delta}^{\Re}$ and $\|\varphi_{\varepsilon}\|_{L_1} = 1$ for any $\varepsilon > 0$, it follows then $A_{\varepsilon} \to 0$ as $\varepsilon \to 0$, i.e., the relation (3.12) is established. Moreover, the formula (3.11) together with (3.12) proves the relation (3.9), and then the theorem. □

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