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EXISTENCE OF INFINITELY MANY LARGE SOLUTIONS FOR A CLASS OF FOURTH-ORDER ELLIPTIC EQUATIONS IN R^N

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Abstract

In this paper, we study the following fourth-order elliptic equations:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x, u), \text{ in } R^N \\ u \in H^2(R^N). \end{cases}$$

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Using the variant fountain theorem, under certain assumptions on V and f, we obtain infinitely many large solutions.

1. Introduction and Preliminaries

Consider the following nonlinear fourth-order elliptic equations:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x, u), \text{ in } R^N \\ u \in H^2(R^N), \end{cases} \tag{1.1}$$

where $N \geq 1$, $V \in C(R^N, R)$, $f \in C(R^N \times R, R)$.

In recent years, the existence or multiplicity of solutions for fourth-order elliptic equations have been widely studied, see, for example, [1 - 13]. Specially, for the case of a bounded domain, there are a number of papers concerned with the equations like or similar to (1.1). For example, An and Liu [2] use the mountain pass theorem to get the existence results; Wang [9] use linking approaches to obtain at least three nontrivial solutions; Yang and Zhang [10] consider the existence of positive, negative, and sign-changing solutions; etc.

There are several authors, who considered the equations like or similar to (1.1) on the whole space R^N. For example, Chabrowski and Marcos do Ó [11] studied the existence of two solutions; Liu et al. [12] use mountain pass theorem to get existence and multiplicity of solutions under the lack of compactness of embedding of the space; Yin and Wu [13] use mountain pass theorem to get the high energy solutions and nontrivial solutions for Equation (1.1) under the following variant "A-R" type condition: There exist $\mu > 2$ and $r > 0$ such that

$$\mu F(x, u) := \mu \int_0^u f(x, t) dt \leq u f(x, u), \tag{$*$}$$

for all $x \in R^N$ and $|u| \geq r$. The condition $(*)$ guaranteed the boundedness of (P.S.) sequences of the corresponding functional.

In the present paper, we will cancel the assumption $(*)$ and use variant fountain theorem to research existence of infinitely many large solutions for Equation (1.1) under the following hypotheses on potential V and nonlinear term f:

(V_1) $\inf_{x \in R^N} V(x) \geq a > 0$ and for any $M > 0$, $meas\{x \in R^N : V(x) \leq M\} < \infty$, where a is a constant and $meas$ denote Lebesgue measure in R^N.

(f_1) $|f(x, u)| \leq C(1 + |u|^{p-1})$ for all $(x, u) \in R^N \times R$, here $p \in (2, 2_*)$, $2_* = \dfrac{2N}{N-4}$ if $N > 4$; $2_* = \infty$ if $N \leq 4$.

(f_2) $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in R^N$.

(f_3) There exists $\alpha > 2$ such that $\liminf_{|u| \to \infty} \dfrac{f(x, u)u}{|u|^{\alpha}} > 0$ uniformly for $x \in R^N$.

(f_4) For a.e. $x \in R^N$, $\dfrac{1}{2} f(x, s)s - F(x, s)$ is increasing in $s > 0$. For a.e. $x \in R^N$, $f(x, u) \geq 0$ for all $u \geq 0$.

(f_5) $f(x, -u) = -f(x, u)$, $\forall (x, u) \in R^N \times R$.

Remark 1.1. There are potentials V do satisfy (V_1), for example, $V(x) = x^2 + 1$. There are functions f satisfying the assumptions (f_1) - (f_5) but not satisfying $(*)$, for example, $f(x, u) = f(u) = u^3 \ln(2|u| + 1)$. Evidently, $f(u) = u^3 \ln(2|u| + 1)$ satisfying the assumptions $(\mathrm{f}_1) - (\mathrm{f}_3)$ and (f_5). Since for all $x \in R^N$, $\dfrac{f(u)}{u} = u^2 \ln(2|u| + 1)$ is an increasing in $u > 0$, then for $u > 0, h(t) := \frac{1}{2}t^2 f(x, u)u - F(x, tu)$ is increasing in $t \in (0, 1]$. This implies the assumption (f_4) be satisfied.

Before stating our main results, we give several notations. Define the function space

$$H = H^2(R^N) := \{u \in L^2(R^N) : |\nabla u|, \Delta u \in L^2(R^N)\},$$

with the inner product and norm

$$\langle u, v\rangle_H = \int_{R^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + uv)dx, \quad \|u\|_H^2 = \langle u, u\rangle_H.$$

Set

$$E := \{u \in H : \int_{R^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx < \infty\},$$

then E is a Hilbert space with the following inner product and the norm:

$$\langle u, v\rangle_E = \int_{R^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\|_E^2 = \langle u, u\rangle_E.$$

Throughout the paper, c_i will denote various positive constants independent of the functions. The main result of the present paper is the following theorem:

Theorem 1.1. *If* (V_1) *and* (f_1) - (f_5) *hold, then the Equation* (1.1) *has infinitely many large nontrivial solutions.*

Remark 1.2. Obviously, it follows from (V_1) that the embedding $E \hookrightarrow L^s(R^N)$ is continuous, for any $s \in [2, 2_*]$. Under the assumption (V_1), motivated by Lemma 3.4 in [14], we can prove that the embedding $E \hookrightarrow L^s(R^N)$ is compact, for any $s \in [2, 2_*)$.

It is well known that a weak of equation (1.1) is a critical point of the following functional:

$$I(u) = \frac{1}{2}\int_{R^N}(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx - \int_{R^N} F(x, u)dx.$$

Under the above assumptions, it is easy to know that $I \in C^1(E, R)$ and

$$\langle I'(u), v\rangle = \int_{R^N}(\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv)dx - \int_{R^N} f(x, u)v dx, \ \forall u, v \in E,$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between E and E^*. Since we do not assume $(*)$, the verification of (P.S.) condition becomes complicated, so we use the following variant fountain theorem introduced in [15] without (P.S.) condition to handle this problem.

Lemma 1.1 (Variant fountain theorem). *Let E be a Banach space with the norm $\|\cdot\|_E$ and $E = \overline{\oplus_{j \in N} X_j}$ with $\dim X_j < \infty$ for any $j \in N$. Set $Y_k = \oplus_{j=0}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$ and*

$$B_k = \{u \in Y_k : \|u\|_E \le \rho_k\}, \ N_k = \{u \in Z_k : \|u\|_E = r_k\} \text{ for } \rho_k > r_k > 0.$$

Consider the following C^1-functional $I_\lambda : E \to R$ defined by:

$$I_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

We assume that

(F_1) I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$.

(F_2) $B(u) \ge 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_E \to \infty$.

Let, for $k \ge 2$,

$$c_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)),$$

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\|_E = r_k} I_\lambda(u),$$

$$a_k(\lambda) := \max_{u \in Y_k, \|u\|_E = \rho_k} I_\lambda(u),$$

where $\Gamma_k := \{\gamma \in C(B_k, E) | \gamma$ is odd, $\gamma|_{\partial B_k} = id\}$. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1, 2]$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in [1, 2]$. Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \|u_n^k(\lambda)\|_E < \infty, \ I'_\lambda(u_n^k(\lambda)) \to 0 \text{ and } I_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \text{ as } n \to \infty.$$

2. Proof of Theorem 1.1

Since $E \hookrightarrow L^2(R^N)$ and $L^2(R^N)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_j\}$. Set $X_j := Re_j$, then define $Y_k = \oplus_{j=0}^k X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. Consider the family of functionals $I_\lambda : E \to R$ defined by

$$I_\lambda(u) := \frac{1}{2}\|u\|_E^2 - \lambda \int_{R^N} F(x, u)dx := A(u) - \lambda B(u),$$

for $\lambda \in [1, 2]$. Then $B(u) \geq 0$ for all $u \in E$, $A(u) \to \infty$ as $\|u\|_E \to \infty$, and $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. And it is easy to see that I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$.

To complete the proof of our theorem, we need the following lemmas:

Lemma 2.1. *For any* $2 < p < 2_*$, *we have that*

$$\beta_k := \sup_{u \in Z_k, \|u\|_E = 1} \|u\|_p \to 0 \text{ as } k \to \infty.$$

Proof. Obviously, the sequence $\{\beta_k\}$ is nonnegative and nonincreasing. Suppose that $\beta_k \to \beta > 0$ as $k \to \infty$. Then for any k sufficiently large, there exists a $u_k \in Z_k$ with $\|u_k\|_E = 1$ and $\|u_k\|_p \geq \frac{\beta}{2}$. For any $u \in E$, since $\{e_j\}$ is an orthogonal basis of E, there exists a sequence

$\{\alpha_j\} \subset R$ satisfying $u = \sum_{j=1}^{\infty} \alpha_j e_j$, thus by the Schwartz inequality and the Parseval equality, we have

$$|\langle u, u_k \rangle_E| = |\langle \sum_{j=1}^{\infty} \alpha_j e_j, u_k \rangle_E| = |\langle \sum_{j=k}^{\infty} \alpha_j e_j, u_k \rangle_E|$$

$$\leq \|\sum_{j=k}^{\infty} \alpha_j e_j\|_E \|u_k\|_E = \sqrt{\sum_{j=k}^{\infty} \alpha_j^2} \to 0 \text{ as } k \to \infty.$$

Therefore, we obtain that $u_k \rightharpoonup 0$ in E and thus, up to a subsequence, $u_k \to 0$ in $L^p(R^N)$ because the embedding $E \hookrightarrow L^p(R^N)$ is compact. This contradiction completes the proof.

Lemma 2.2. *If* (V_1) *and* (f_1) - (f_3) *hold, then there exist* $\lambda_n \to 1$ *as* $n \to \infty$, $\bar{c}_k > \bar{b}_k > 0$ *and* $\{u_n\} \subset E$ *such that* $I'_{\lambda_n}(u_n) = 0$, $I_{\lambda_n}(u_n) \in [\bar{b}_k, \bar{c}_k]$.

Proof. (i) By (f_1) - (f_3), we know that there are positive constants $c_1 > 0$, $c_2 > 0$ such that

$$F(x, u) \geq c_1 |u|^\alpha - c_2 u^2, \quad \forall (x, u) \in R^N \times R.$$

Hence, for all $u \in Y_k$,

$$I_\lambda(u) = \frac{1}{2}\|u\|_E^2 - \lambda \int_{R^N} F(x, u) dx$$

$$\leq \frac{1}{2}\|u\|_E^2 - \lambda c_1 \|u\|_\alpha^\alpha + \lambda c_2 \|u\|_2^2$$

$$\leq \frac{1}{2}\|u\|_E^2 - c_3 \|u\|_E^\alpha + c_4 \|u\|_E^2,$$

where in the last inequality, we use the equivalence of all norms on the finite dimensional subspace Y_k. Then, we can choose $\|u\|_E = \rho_k > 0$ large enough such that

$$a_k(\lambda) = \max_{u \in Y_k, \|u\|_E = \rho_k} I_\lambda(u) \le 0.$$

(ii) By (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $x \in R^N$, $u \in R$,

$$F(x, u) \le \varepsilon |u|^2 + C_\varepsilon |u|^p.$$

Hence, for any $u \in Z_k$ and $\varepsilon > 0$ small enough

$$I_\lambda(u) \ge \frac{1}{2}\|u\|_E^2 - \lambda\varepsilon\|u\|_2^2 - \lambda C_\varepsilon \|u\|_p^p$$

$$\ge \left(\frac{1}{2} - \frac{\lambda\varepsilon}{a}\right)\|u\|_E^2 - \lambda C_\varepsilon \beta_k^p \|u\|_E^p,$$

where a is a lower bound of $V(x)$ from (V_1) and β_k is defined in Lemma 2.1. Choosing $r_k = (\lambda C_\varepsilon p \beta_k^p)^{\frac{1}{2-p}}$, then

$$b_k(\lambda) = \inf_{u \in Z_k, \|u\|_E = r_k} I_\lambda(u)$$

$$\ge \inf_{u \in Z_k, \|u\|_E = r_k} \left[\left(\frac{1}{2} - \frac{\lambda\varepsilon}{a}\right)\|u\|_E^2 - \lambda C_\varepsilon \beta_k^p \|u\|_E^p\right]$$

$$\ge \left(\frac{1}{2} - \frac{\lambda\varepsilon}{a} - \frac{1}{p}\right) r_k^2$$

$$:= \overline{b}_k.$$

Since $\beta_k \to 0$ as $k \to \infty$ and $p > 2$, for small enough ε, we have $b_k(\lambda) \ge \overline{b}_k \to \infty$ as $k \to \infty$ uniformly for λ. Therefore, by Lemma 1.1, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sup_u \|u_n^k(\lambda)\|_E < \infty, \; I'_\lambda(u_n^k(\lambda)) \to 0 \quad \text{and} \quad I_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \ge b_k(\lambda) \ge$$

$\overline{b}_k$ as $n \to \infty$, where $c_k(\lambda)$ is defined in Lemma 1.1. Furthermore, since

$c_k(\lambda) \leq \sup_{u \in B_k} I(u) := \bar{c}_k$ and E is imbedded compactly to $L^s(R^N)$ for any $s \in [2, 2_*)$, by standard argument, $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ has a convergent subsequence. Hence, there exists $u^k(\lambda)$ such that $I'_\lambda(u^k(\lambda)) = 0$ and $I_\lambda(u^k(\lambda)) \in [\bar{b}_k, \bar{c}_k]$, for a.e. $\lambda \in [1, 2]$. So, when $\lambda_n \to 1$, with $\lambda_n \in [1, 2]$, we find a sequence $\{u^k(\lambda_n)\}$ (denoted by u_n for simplicity) satisfying $I'_{\lambda_n}(u_n) = 0$, $I_{\lambda_n}(u_n) \in [\bar{b}_k, \bar{c}_k]$. This completes the proof.

Lemma 2.3. *Under the assumptions of Theorem* 1.1, *the sequence* $\{u_n\}$ *is bounded.*

Proof. We suppose that $\|u_n\|_E \to \infty$ as $n \to \infty$. Consider $w_n := \frac{u_n}{\|u_n\|_E}$.

Then, up to a subsequence, we obtain

$$w_n \rightharpoonup w \text{ in } E,$$

$$w_n \to w \text{ in } L^s(R^N) \text{ for any } s \in [2, 2_*),$$

$$w_n(x) \to w(x) \text{ a.e. } x \in R^N.$$

Case 1. Suppose $w \neq 0$ in E. By $I'_{\lambda_n}(u_n) = 0$, we have

$$0 = \langle I'_{\lambda_n}(u_n), u_n \rangle = \|u_n\|_E^2 - \lambda_n \int_{R^N} f(x, u_n) u_n dx.$$

Therefore, there exists a constant $c_5 > 0$ such that

$$\int_{R^N} \frac{f(x, u_n)u_n}{\|u_n\|_E^2} dx \leq c_5.$$

On the other hand, by Fatou's lemma, we have

$$\liminf_{n \to \infty} \int_{R^N} \frac{f(x, u_n)u_n}{\|u_n\|_E^2} dx \geq \int_{R^N} \liminf_{n \to \infty} \frac{f(x, u_n)u_n}{\|u_n\|_E^2} dx$$

$$= \int_{R^N} \liminf_{n\to\infty} |w_n|^2 \frac{f(x, u_n)u_n}{|u_n|^2} dx$$

$$= \infty.$$

This is a contradiction.

Case 2. Suppose $w = 0$ in E. Inspired by [16], we define

$$I_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} I_{\lambda_n}(tu_n).$$

For any $c > 0$, let $\overline{w}_n := \sqrt{4c}w_n$. Since for all $x \in R^N$, $u \in R$, $F(x, u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^p$, we get

$$\int_{R^N} F(x, \overline{w}_n)dx \leq \varepsilon\int_{R^N} |\overline{w}_n|^2 dx + C_\varepsilon\int_{R^N} |\overline{w}_n|^p dx \to 0.$$

Then, for n large enough, we have

$$I_{\lambda_n}(t_n u_n) \geq I_{\lambda_n}(\overline{w}_n) = 2c - \lambda_n\int_{R^N} F(x, \overline{w}_n)dx \geq c,$$

which implies that $\lim_{n\to\infty} I_{\lambda_n}(t_n u_n) = \infty$. Evidently, $t_n \in (0, 1)$, we know that $\langle I'_{\lambda_n}(t_n u_n), t_n u_n\rangle = 0$. Thus, by conditions (f_4) and (f_5), we obtain

$$I_{\lambda_n}(t_n u_n) = I_{\lambda_n}(t_n u_n) - \frac{1}{2}\langle I'_{\lambda_n}(t_n u_n), t_n u_n\rangle_E$$

$$= \lambda_n\int_{R^N} [\frac{1}{2} f(x, t_n u_n)t_n u_n - F(x, t_n u_n)]dx$$

$$\leq \lambda_n\int_{R^N} [\frac{1}{2} f(x, u_n)u_n - F(x, u_n)]dx$$

$$= I_{\lambda_n}(u_n) - \frac{1}{2}\langle I'_{\lambda_n}(u_n), u_n\rangle_E$$

$$= I_{\lambda_n}(u_n) \in [\overline{b}_k, \overline{c}_k].$$

This contradiction completes the proof.

Proof of Theorem 1.1. Combining Lemmas 2.2 and 2.3, since $E \hookrightarrow L^s(R^N)$, $2 < s < 2_*$ is compact, standard argument implies that, up to a subsequence, $u_n \to u^k$ in E as $n \to \infty$. By $\{u_n\} \subset E$ is bounded, we have $\int_{R^N} F(x, u_n)dx$ is bounded. Therefore, by $I_{\lambda_n}(u_n) \in [\bar{b}_k, \bar{c}_k]$ and

$$I(u_n) = I_{\lambda_n}(u_n) + (\lambda_n - 1)\int_{R^N} F(x, u_n)dx,$$

we obtain $I(u^k) = \lim_{n\to\infty} I(u_n) \in [\bar{b}_k, \bar{c}_k]$. By $I'_{\lambda_n}(u_n) = 0$ and

$$\langle I'(u_n), v\rangle = \langle I'_{\lambda_n}(u_n), v\rangle - (\lambda_n - 1)\int_{R^N} f(x, u_n)vdx, \text{ for all } v \in E,$$

we know that $I'(u_n) \rightharpoonup 0$ in E^* as $n \to \infty$. Since $I \in C^1(E)$, we have $I'(u_n) \to I'(u^k)$ in E^* as $n \to \infty$. This means $I'(u^k) = 0$. By $I_{\lambda_n}(u_n) \in [\bar{b}_k, \bar{c}_k]$ and $\bar{b}_k \to \infty$ as $n \to \infty$, we know that $\{u^k\}_{k=1}^{\infty}$ is an unbound sequence of critical points of functional I. This completes the proof.

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