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ON THE STABILITY OF VECTOR VALUED PEXIDEREZED FUNCTIONAL EQUATIONS

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Abstract

In this paper, we first study the superstability of the Wilson's equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \tag{1}$$

for normed algebra, with multiplicative norm, valued functions on semigroups and secondly we investigate the stability problem of Pexiderezed cosine functional equations for vector valued functions.

1. Introduction

Let S be a semigroup, E be a normed algebra with multiplicative norm and σ denotes an involution of S, i.e., $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$, for all $x, y \in S$. It has been proved in [2] that the functional equation

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$$f(xy) = f(x)f(y), \quad \text{for all } x, y \in S, \tag{1.1}$$

is superstable in the class of functions $f : S \to E$, i.e., every such function satisfying the inequality

$$\|f(xy) - f(x)f(y)\| \le S, \quad \text{for all } x, y \in S,$$

where δ is a fixed positive real number either is bounded or satisfies (1.1).

Similar results of that kind have been established for various type of functional equations, for more details, see [1], [2], [3], [4], [9], [10].

In [9], Roukbi et al. proved the superstability of the Wilson's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad \text{for all } x, y \in G, \tag{1.2}$$

where G is any group and f, g are complex-valued functions on G.

In the first part of this paper, we shall extend Baker's result in his paper [2] from (1.1) to (1.2) on semigroup S.

Now, let G be a group and A be a complex normed algebra with identity.

We recall that a vector valued function f on a group G is said to satisfy Kannappan's condition (K), if

$$f(xyz) = f(xzy), \quad \text{for all } x, y, z \in G.$$

In [5], Kannappan and Kim studied the stability of the generalized cosine functional equations

$$f(x + y) + f(x - y) = 2f(x)g(y), \quad \text{for all } x, y \in G, \tag{A}$$

and

$$f(x + y) + f(x - y) = 2g(x)f(y), \quad \text{for all } x, y \in G, \tag{B}$$

where $f, g : G \to A$ such that f satisfies the Kannappan's condition (K).

In the second part, we investigate an approximation result of the functional equations (A) and (B) with a general involution. More precisely, we study the stability of the following functional equations:

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad \text{for all } x, y \in S, \tag{C}$$

and

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad \text{for all } x, y \in S. \tag{D}$$

Our contribution in this part is to remove the restriction (K) and the norm of algebra here need not be multiplicative. For more informations concerning stability of functional equations, we refer to [1-12].

2. Stability of the Equation (1)

The following lemma will be used in Proof of Theorem 2.2.

Lemma 2.1. *Let* $\delta > 0$ *be given*, *S be a semigroup*, *and E be a normed algebra with multiplicative norm*. *Assume that functions* $f, g : S \to E$ *satisfy the inequality*

$$\|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\| \leq \delta, \quad \textit{for all } x, y \in S, \tag{2.1}$$

such that $f \neq 0$. *If g is unbounded*, *then so is f.*

Proof. Assume that g is unbounded function satisfying the inequality (2.1). If $f \neq 0$ is bounded, let $M = \sup\|f(x)\|, x \in S$ and choose $a \in S$ such that $f(a) \neq 0$, then we get from the inequality (2.1) that

$$\|f(ay) + f(a\sigma(y)) - 2f(a)g(y)\| \leq \delta, \quad \text{for all } y \in S,$$

from which we obtain that

$$\|2f(a)g(y)\| - \|f(ay) + f(a\sigma(y))\| \leq \delta, \quad \text{for all } y \in S,$$

so, using the fact that the norm of E is multiplicative, we obtain that

$$\|g(y)\| \leq \frac{\delta + 2M}{2\|f(a)\|}, \quad \text{for all } y \in S,$$

then g is bounded, which contradicts our assumption. □

In Theorem 2.2 below, the superstability result of the Equation (1.1) will be investigated on any semigroup.

Theorem 2.2. *Let* $\delta > 0$ *be given*, S *be a semigroup*, *and* E *be a normed algebra with multiplicative norm*. *Assume that the functions* $f, g : S \to E$ *satisfy the inequality*

$$\|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\| \leq \delta, \tag{2.2}$$

for all $x, y \in S$. *Then one of the following holds*:

(i) f, g *are bounded*;

(ii) *f is unbounded and g satisfies the equation*

$$g(xy) + g(yx) + g(x\sigma(y)) + g(\sigma(y)x) = 4g(x)g(y), \text{ for all } x, y \in S; \tag{2.3}$$

(iii) *g is unbounded and the pair* (f, g) *satisfies the Equation* (1).

Proof. (ii) Assume that f, g satisfy (2.2). First, we consider the case of f unbounded. For all $x, y, z \in S$, using the fact that the norm of E is multiplicative, we have

$$\begin{aligned}
&2\|f(z)\|\,\|g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) - 4g(x)g(y)\| \\
&= \|2f(z)g(xy) + 2f(z)g\|\,(x\sigma(y)) + 2f(z)g(yx) \\
&\quad + 2f(z)g(\sigma(y)x) - 8f(z)g(x)g(y)\,\| \\
&\leq \|f(zxy) + f(z\sigma(y)\sigma(x)) - 2f(z)g(xy)\| \\
&\quad + \|f(zx\sigma(y)) + f(zy\sigma(x)) - 2f(z)g(x\sigma(y))\| \\
&\quad + \|f(zyx) + f(z\sigma(x)\sigma(y)) - 2f(zx)g(y)\| \\
&\quad + \|f(z\sigma(y)x) + f(z\sigma(x)y) - 2f(z)g(\sigma(y)x)\| \\
&\quad + \|f(zxy) + f(zx\sigma(y)) - 2f(z)g(yx)\|
\end{aligned}$$

$$+ \|f(zyx) + f(zy\sigma(x)) - 2f(zy)g(x)\|$$

$$+ \|f(z\sigma(y)x) + f(z\sigma(y)\sigma(x)) - 2f(z\sigma(y))g(x)\|$$

$$+ \|f(z\sigma(x)y) + f(z\sigma(x)\sigma(y) - 2f(z\sigma(x))g(y)\|$$

$$+ 2\|g(y)\| \|f(zx) + f(z\sigma(x)) - 2f(z)g(x)\|$$

$$+ 2\|g(x)\| \|f(zy) + f(z\sigma(y) - 2f(z)g(y)\|.$$

By virtue of the inequality (2.2), we get that

$$2\|f(z)\| \|g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) - 4g(x)g(y)\|$$

$$\leq 8\delta + 2(\|g(x)\| + \|g(y)\|). \tag{2.4}$$

Since f is unbounded, from (2.4) we conclude that g is a solution of the Equation (2.3), which ends the proof in this case.

(iii) Now suppose that g is unbonded. For $f = 0$ the pair (f, g) is a trivial solution of the Equation (1). Assume that $f \neq 0$. For all $x, y, z \in S$, we have

$$2\|g(z)\| \|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\|$$

$$= \|2f(xy)g(z) + 2f(x\sigma(y))g(z) - 4f(x)g(y)g(z)\|$$

$$\leq \|f(xyz) + f(xy\sigma(z)) - 2f(xy)g(z)\|$$

$$+ \|f(x\sigma(y)z) + f(x\sigma(y)\sigma(z)) - 2f(x\sigma(y))g(z)\|$$

$$+ \|f(xyz) + f(x\sigma(z)\sigma(y)) - 2f(x)g(yz)\|$$

$$+ \|f(xy\sigma(z)) + f(xz\sigma(y)) - 2f(x)g(y\sigma(z))\|$$

$$+ \|f(x\sigma(y)z) + f(x\sigma(z)y) - 2f(x)g(\sigma(z)y)\|$$

$$+ \|f(x\sigma(y)\sigma(z)) + f(xyz) - 2f(x)g(zy)\|$$

$$+ \|f(x\sigma(z)y) + f(x\sigma(z)\sigma(y)) - 2f(x\sigma(z))g(y)\|$$

$$+ \|f(xyz) + f(xz\sigma(y)) - 2f(xz)g(y)\|$$

$$+ \| 2f(x)g(yz) + 2f(x)g(y\sigma(z)) + 2f(x)g(\sigma(z)y)$$

$$+ 2f(x)g(zy) - 8f(x)g(y)g(z) \|$$

$$+ \|2f(xz)g(y) + 2f(x\sigma(z))g(y) - 4f(x)g(y)g(z)\|.$$

In virtue of inequality (2.2), we obtain

$$2\|g(z)\| \|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\|$$

$$\leq 8\delta + 2\delta\|g(y)\| + \|2f(x)\| \|g(yz) + g(y\sigma(z)) + g(zy) + g(\sigma(z)y) - 4g(y)g(z)\|.$$

In view Lemma 2.1, we see that g is unbounded implies necessarily that f is unbounded hence according to Theorem 2.2 (ii), g is a solution of the Equation (2.3).

So, we conclude that

$$2\|g(z)\| \|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\| \leq 8\delta + 2\delta\|g(y)\|. \tag{2.5}$$

Since g is unbounded, from (2.5), we conclude that the pair (f, g) satisfies the Equation (1), which finished the proof of the Theorem 2.2. □

As a consequence of Theorem 2.2, we have the following result on the superstabilty of the algebra valued d'Alembert's functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad \text{for all } x, y \in S.$$

Corollary 2.3. *Let $\delta > 0$ be given, S be a semigroup, and E be a normed algebra with multiplicative norm. Assume that the function $f : S \to E$ satisfies the inequality*

$$\|f(xy) + f(x\sigma(y)) - 2f(x)f(y)\| \leq \delta, \quad \text{for all } x, y \in S. \tag{2.6}$$

Then either

$$\|f(x)\| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad \text{for all } x \in S,$$

or

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad \text{for all } x, y \in S.$$

Proof. Assume that f satisfies the inequality (2.6). If f is bounded, let $A = \sup\|f(x)\|$. By using (2.6) and that the norm of E is multiplicative, we get that

$$\|2f(x)f(x)\| \leq \delta + 2A, \quad \text{for all } x \in S,$$

from which we obtain that $2A^2 - 2A - \delta \leq 0$ such that

$$A \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad \text{for all } x \in S,$$

the rest of proof is an immediate consequence of Theorem 2.2. □

Corollary 2.4 ([2]). *Let* $\delta > 0$ *be given*, S *be a semigroup*, *and* E *be a normed algebra with multiplicative norm. Assume that the function* $f : S \to E$ *satisfies the inequality*

$$\|f(xy) - f(x)f(y)\| \leq \delta, \quad x, y \in S. \tag{2.6}$$

Then either

$$\|f(x)\| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad x \in S,$$

or

$$f(xy) = f(x)f(y), \quad x, y \in S.$$

Proof. The proof follows from Theorem 2.2 by putting $\sigma = Id$ and $g = f$. □

3. Stability of the Equations (C) and (D)

Theorem 3.1. *Let* S *be a semigroup and* A *be an arbitrary algebra. Suppose that* $f, g : S \to A$ *satisfy the inequality*

$$\|f(xy) + f(x\sigma(y)) - 2g(x)f(y)\| \leq \epsilon, \quad \textit{for all } x, y \in S, \tag{3.1}$$

with f satisfying

$$\|f(x) - f(\sigma(x))\| \leq \eta_1, \quad \textit{for all } x \in S, \tag{3.2}$$

for some $\epsilon, \eta_1 \geq 0$. *Suppose that there is a* $z_0 \in S$ *such that* $g(z_0)$ *invertible and*

$$\|f(z_0 x) + f(z_0 \sigma(x))\| \leq \eta_2, \tag{3.3}$$

for $\eta_2 \geq 0$. *Then there is* $m : S \to A$ *such that*

$$\|m(xy) - m(x)m(y)\| \leq a_1, \quad \textit{for all } x, y \in S, \tag{3.4}$$

and

$$\left\| f(x) - \frac{1}{2}(m(x) + m(\sigma(x))) \right\| \leq a_2, \quad \textit{for all } x \in S, \tag{3.5}$$

for some constants $a_1 \geq 0$ *and* $a_2 \geq 0$.

Proof. Using (3.1) and (3.3), we get that for all $x \in S$

$$\|f(z_0 x) + f(z_0 \sigma(x)) - 2g(z_0)f(x)\| \leq \epsilon$$

$$\|2g(z_0)f(x)\| \leq \epsilon + \|f(z_0 x) + f(z_0 \sigma(x))\|$$

$$\|2g(z_0)f(x)\| \leq \epsilon + \eta_2$$

$$\|g(z_0)f(x)\| \leq \frac{1}{2}(\epsilon + \eta_2).$$

Define the function $h : S \to A$ given by

$$h(x) = \frac{1}{2}(f(x) + f(\sigma(x))), \quad x \in S.$$

Then h is σ-even, that is, $h(x) = h(\sigma(x))$, $x \in S$, and by (3.2), we have

$$\|h(x) - f(x)\| \leq \eta_1 / 2, \quad \|g(z_0)h(x)\| \leq M, \quad \text{for all } x \in S, \tag{3.6}$$

where $M = \epsilon + \eta_2$. Define the function $m : S \to A$ given by

$$m(x) = g(z_0) + h(x), \quad x \in S.$$

Utilizing (3.6), we get that for all $x \in S$

$$\|m(xy) - m(x)m(y)\| = \|g(z_0) + h(xy) - g(z_0)^2 - g(z_0)h(y) - h(x)g(z_0) - h(x)h(y)\|$$

$$\leq \|h(xy)\| + \|h(x)h(y)\| + \|g(z_0)h(y)\| + \|h(x)g(z_0)\| + \|g(z_0)\|^2 + \|g(z_0)\|$$

$$\leq \|h(xy) - f(xy)\| + \|f(xy)\| + \|g(z_0)^{-1}g(z_0)h(x)g(z_0)^{-1}g(z_0)h(y)\|$$

$$+ \|g(z_0)h(y)\| + \|g(z_0)^{-1}g(z_0)h(x)g(z_0)\| + \|g(z_0)\|^2 + \|g(z_0)\|$$

$$\leq \frac{\eta_1}{2} + M\|g(z_0)^{-1}\| + M^2\|g(z_0)^{-1}\|^2 + M$$

$$+ M\|g(z_0)^{-1}\|\|g(z_0)\| + \|g(z_0)\|^2 + \|g(z_0)\|,$$

which gives (3.4) with

$$a_1 := \frac{\eta_1}{2} + M\|g(z_0)^{-1}\|(1 + M\|g(z_0)^{-1}\| + \|g(z_0)\|) + M + \|g(z_0)\|^2 + \|g(z_0)\|.$$

Finally by (3.6), we have for all $x \in S$ that

$$\|f(x) - \frac{1}{2}(m(x) + m(\sigma(x))\| = \|f(x) - h(x) + h(x) - \frac{1}{2}(h(x) + h(\sigma(x)) - g(z_0)\|$$

$$\leq \|f(x) - h(x)\| + \|g(z_0)\|$$

$$\leq \frac{\eta_1}{2} + \|g(z_0)\|,$$

which gives (3.5) with $a_2 := \frac{\eta_1}{2} + \|g(z_0)\|$. This proves the theorem. □

We complete this paper by the following theorem:

Theorem 3.2. *Let S be a semigroup and A be an arbitrary algebra. Suppose that $f, g : S \to A$ satisfy the inequality*

$$\|f(xy) + f(x\sigma(y)) - 2f(x)g(y)\| \leq \epsilon, \quad \text{for all } x, y \in S,$$

with f satisfying

$$\|f(x) - f(\sigma(x))\| \leq \eta_1, \quad \text{for all } x \in S,$$

for some nonnegative ϵ *and* η_1. *Suppose there is a* $z_0 \in S$ *such that* $g(z_0)$ *invertible and*

$$\|f(z_0x) + f(z_0\sigma(x))\| \leq \eta_2 \quad \text{for } \eta_2 \geq 0.$$

Then there exists a mapping $m : S \to A$ *such that*

$$\|m(xy) - m(x)m(y)\| \leq a_1, \quad \text{for all } x, y \in S,$$

and

$$\|f(x) - \frac{1}{2}(m(x) + m(\sigma(x))\| \leq a_2, \quad \text{for all } x \in S,$$

for some constants $a_1 \geq 0$ *and* $a_2 \geq 0$.

The proof runs paralleled to that of Theorem 3.1.

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