

# **NEW FORWARD EQUATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE NOISE**

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## **Abstract**

The existing forward equation defines a probability current that disagrees with the random paths. A new one is adequate. With respect to that, the existing Ito paths consist of most probable increments (rather than mean ones), with Gaussian deviations. Further new features are the compliance with prediction theory and the agreement with physical steady states. Multiplicative noise, in particular the “spurious drift”, does no longer show up in the relevant equations.

## **1. Introduction**

Stochastic differential equations (SDEs) give rise to some problems when the coupling with the standard noise sources (Wiener processes) depends on the random state (“multiplicative noise”) [1-6]. The stochastic integration is then only defined up to a parameter  $\alpha$  ( $0 \leq \alpha \leq 1$ , 0 for

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Ito,  $1/2$  for Stratonovich and  $1$  for “anti-Ito”), and attempts to specify  $\alpha$  in physics met with inconsistencies (the “Ito or Stratonovich dilemma” [7]). It will now be shown that the reason is mathematical: short-time solutions of the forward or Fokker-Planck equation (FPE) are not Gaussian, and the path increments must be related with the most probable value rather than with the mean. This is required for integrating the increments in the mean square sense.

The reasoning consists of several steps. Inspection of the probability current at a density maximum (transient or steady) first shows the need for supplying the existing random increments by an extra term (occurring in the FPE). Formally, this amounts to replacing  $\alpha$  by  $\alpha - 1$  in the SDE, while the FPE is unchanged. The next step concerns the fact that SDE yields Gaussian path increments in  $dt$ . This disagrees with the oblique solution of the FPE, but for the (mean square) integration of the path increments it is sufficient to know the Gaussian that fits the oblique density at its maximum. That substitute is given by the SDE when  $\alpha = 1$ , i.e., for the anti-Ito FPE, corresponding to the Ito SDE. This is the only possibility. The Ito increments are thereby centered at their most probable value.

The optimum prediction now agrees with the general theory [1, 3]; it is the expectation, determined by the noiseless motion. The anti-Ito FPE describes the equilibrium states which are important in physics. It is a remarkable new finding that these are approached and determined by the Ito paths (with the mean, i.e., the noiseless motion, tending to the density maximum).

The above results are confirmed by a tensor argument. The “spurious drift” arising with multiplicative noise is not a tensor. The request of compatibility with the tensor laws excludes its occurrence. This readily imposes the Ito sense of the SDE, and among the equilibrium states it singles out those of the anti-Ito FPE.

As usual, random values will be denoted by uppercase and fixed ones by lowercase letters.

## 2. Background

### 2.1. Generalities

The continuous Markov process  $\vec{X}(t)$  is supposed to obey the SDE

$$dX^i = a^i(\vec{X})dt + b^{ik}(\vec{X})dW_k \quad \text{or} \quad d\vec{X} = \vec{a}(\vec{X})dt + \underline{B}(\vec{X})d\vec{W}, \quad (2.1)$$

with smooth functions  $a^i(\vec{X})$ ,  $b^{ik}(\vec{X})$ . As usual, (2.1) denotes an integral equation, with a “sense” specified by  $\alpha$  ( $0 \leq \alpha \leq 1$ ). The (independent) Wiener processes  $W_k(t)$  obey  $\langle W_k(t) - W_k(0) \rangle = 0$  and  $\langle [W_k(t) - W_k(0)]^2 \rangle = t$ .

The existing expression for the increments, with given  $\vec{X}(t) = \vec{x}$  and  $dt \geq 0$ , is

$$\vec{X}(t + dt) - \vec{x} = \vec{a}(\vec{x})dt + \underline{B}(\vec{x})d\vec{W} + \alpha \vec{a}_{Sp}(\vec{x})dt + o(dt), \quad (2.2)$$

see [1-6], where  $d\vec{W} := \vec{W}(t + dt) - \vec{W}(t)$ , and with the “spurious” drift

$$\alpha^i_{Sp}(\vec{x}) := b^{ij},_{,k}(\vec{x})b^{kj}(\vec{x}) = (\underline{B},_k \underline{B}^T)^{ik}. \quad (2.3)$$

The time evolution of the probability density  $w(\vec{x}, t)$  of  $\vec{X}(t)$  is determined by the Fokker-Planck equation [1-6]. Its “drift” is given by the expectation of (2.2)

$$\langle \vec{X}(t + dt) - \vec{x} \rangle = [\vec{a}(\vec{x}) + \alpha \vec{a}_{Sp}(\vec{x})]dt + o(dt), \quad (2.4)$$

more precisely by

$$\vec{a}(\vec{x}) + \alpha \vec{a}_{Sp}(\vec{x}), \quad (2.5)$$

and the “diffusion matrix” by

$$\underline{D}(\bar{x}) = \underline{B}(\bar{x})\underline{B}^T(\bar{x}). \quad (2.6)$$

The explicit FPE reads

$$w_{,t} = [ -(\alpha^i + \alpha\alpha^i_{Sp})w + (1/2)(D^{ik}w)_{,k} ],_i. \quad (2.7)$$

When  $\underline{B}$  is diagonal (as, e.g., in one dimension), the spurious drift is determined by  $\underline{D}$

$$\alpha^i_{Sp} = D^{ik}_{,k} / 2. \quad (2.8)$$

This expression always holds when  $\bar{\alpha}_{Sp}$  is supposed to vanish when  $\underline{D}(\bar{x})$  is constant, as it will be assumed in what follows.

### A short proof:

A general  $\underline{B}$  can be written as  $\sqrt{\underline{D}}\underline{O}$  with an orthogonal  $\underline{O}$ . Setting  $\sqrt{\underline{D}}\underline{O}d\bar{W} := \sqrt{\underline{D}}d\tilde{W}$  defines the new Wiener increments  $\underline{O}(\bar{x})d\bar{W} := d\tilde{W}$ , which for each  $\bar{x}$  are stochastically equivalent with  $d\bar{W}$  [1]. This eliminates  $\underline{O}$ , and (2.3) becomes  $[(\sqrt{\underline{D}})_{,k}\sqrt{\underline{D}}]^{ik}$ , which equals (2.8) by diagonalizing at each  $\bar{x}$ . This is but the simplest version: each orthogonal  $\underline{Q}(\bar{x})$  defines a possible  $d\tilde{W} := \underline{Q}(\bar{x})d\bar{W}$ , but only  $\underline{Q} = \underline{O}$  yields a zero  $\bar{\alpha}_{Sp}$  when  $\underline{D}$  is constant. For the full variety of  $\bar{\alpha}_{Sp}$ , see [8].

### 2.2. The “noise-generated drift”

By  $(D^{ik}w)_{,k} = D^{ik}_{,k}w + D^{ik}w_{,k}$ , and by exhibiting the “noise-generated drift”

$$\alpha^i_{NG} := - (1/2)D^{ik}_{,k}, \quad (2.9)$$

the FPE can be rewritten in the form

$$w_{,t} = \nabla \cdot [ - (\bar{a} + \alpha \bar{a}_{Sp} + \bar{a}_{NG})w + \underline{D}\nabla w / 2 ].$$

Clearly  $\bar{a}_{NG}$  originates from the unequal expulsion by diffusion at neighbouring sites (the effect of thermodiffusion [9]). It is absent in the SDE, and mainly also in the *path increments* (2.2) and (2.4), although its impact is a real phenomenon. By (2.8), it follows that

$$\bar{a}_{NG} = - \bar{a}_{Sp}, \quad (2.10)$$

and the FPE assumes the form

$$w_{,t} = \nabla \cdot \{ - [\bar{a} + (\alpha - 1)\bar{a}_{Sp}]w + \underline{D}\nabla w / 2 \}. \quad (2.11)$$

With the probability current

$$\vec{J} := [\bar{a} + (\alpha - 1)\bar{a}_{Sp}]w - \underline{D}\nabla w / 2, \quad (2.12)$$

the FPE reduces to the continuity equation  $w_{,t} + \nabla \cdot \vec{J} = 0$  expressing the impossibility of a spontaneous birth and death of the paths.

The “propagator”  $g(\vec{x}, t, \vec{x}_0)$ ,  $t \geq 0$ , is the solution of (2.11) with  $g(\vec{x}, 0, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0)$ , asymptotically for  $t \rightarrow 0$ .

### 3. The New Path Increments

Consider any (transient) smooth maximum of the density  $w(\vec{x}, t)$ . There the current (2.12) is  $\vec{J} = [\bar{a} + (\alpha - 1)\bar{a}_{Sp}]w$ , while the mean increment (2.4) always equals  $\langle d\vec{X} \rangle = (\bar{a} + \alpha \bar{a}_{Sp})dt$ . This shows that (2.2) and (2.4) must be *completed* by  $-\bar{a}_{Sp}dt = \bar{a}_{NG}dt$  (while (2.1) remains unchanged). The resulting new random increment is

$$\vec{X}(t + dt) - \vec{x} = \bar{a}(\vec{x})dt + \underline{B}(\vec{x})d\vec{W} + (\alpha - 1)\bar{a}_{Sp}(\vec{x})dt + o(dt) := \Delta\vec{X}. \quad (3.1)$$

This reduces  $\alpha$  by 1 for the paths, but not for the FPE. The Ito increment

$$\Delta\bar{X} = \bar{a}(\bar{x})dt + \underline{B}(\bar{x})d\bar{W}, \quad (3.2)$$

belongs thus to the FPE

$$w_{,t} = \nabla \cdot (-\bar{a}w + \underline{D}\nabla w / 2), \quad (3.3)$$

with the current

$$\bar{J} = \bar{a}w - \underline{D}\nabla w / 2. \quad (3.4)$$

Clearly the paths given by (3.2) follow the noiseless motion in the mean.

#### 4. The Propagators and their Maxima

Explicit solution for time-dependent FPEs are only available for  $\alpha = 1/2$ . The maxima of further propagators can be obtained by specific methods. For simplicity this will now be shown in one dimension, with  $a \equiv 0$  and for the initial value  $x_0 = 0$ .

(a)  $\alpha = 1/2$

The FPE  $w_{,t} = (1/2)[-(bb'w) + (b^2w)'] (D = b^2, \alpha_{Sp} = bb')$  is exactly solved by

$$w(x, t) = b^{-1}(x)(2\pi t)^{-1/2} \exp(-z^2/2t), \quad \text{where} \quad z(x) = \int_0^x b^{-1}(\xi)d\xi. \quad (4.1)$$

The maximum point is given by  $-b^{-1}b'w - wzt^{-1}z' = 0$ , which by  $z' = b^{-1}$  reduces to  $b' + z/t = 0$ . With  $z \approx x/b$  for small  $dt$  this results in

$$\hat{x} \approx -b'bd t = a_{NG}dt. \quad (4.2)$$

This differs from the mean  $b'bd t / 2 = \alpha_{Sp}dt / 2$ , even by the sign. The propagator is thus oblique for each  $dt > 0$ .

(b)  $\alpha = 1$

The FPE reads  $w_{,t} = (Dw')' / 2 = (D'w' + Dw'') / 2$ . At the maximum of  $w$  this reduces to  $w_{,t} = Dw'' / 2$ , which does not involve any derivatives of  $D$ . The result is thus locally same as for a constant  $D$ , i.e.,

$$g(x, dt, 0) \approx (2\pi dt)^{-1/2} D^{-1/2}(0) \exp\{-x^2 / [2D(0)dt]\}, \quad (4.3)$$

a Gaussian centered at the starting point. The full propagator is the oblique density

$$g(x, dt, 0) = (2\pi dt)^{-1/2} D^{-1/2}(x) \exp\{-[x - \alpha_{Sp}(0)dt]^2 / [2D(0)dt]\}. \quad (4.4)$$

With  $D(x) \approx D(0) + xD'(0) \approx D(0) \exp[xD'(0) / D(0)]$  it is easily seen that (4.3) is recovered, but the tails of (4.4) are asymmetric and yield the well-known mean  $\alpha_{Sp}(0)dt$ . The further evolution is determined by  $J = -Dw' / 2$ , which shows that the maximum stays at  $x = 0$ .

With a non-singular  $\underline{D}$  (4.4) can be generalized to

$$g(\bar{x}, dt, \bar{0}) = (2\pi dt)^{-n/2} [\det \underline{D}(\bar{x})]^{-1/2} \\ \times \exp\{-[\bar{x} - \bar{\alpha}_{Sp}(\bar{0})dt]^T \underline{D}^{-1}(\bar{0})[\bar{x} - \bar{\alpha}_{Sp}(\bar{0})dt] / 2dt\}.$$

The full solution with an initial delta-function will be discussed below in the Chapter 6.

(c)  $\alpha = 0$

The FPE is  $w_{,t} = (Dw)'' / 2$ , and the substitution  $Dw := u$  results in  $u_{,t} = Du'' / 2$ . Near  $x = 0$  this is solved by  $(2\pi dt)^{-1/2} D^{-1/2}(0) \exp\{-x^2 / [2D(0)dt]\}$ , and by  $w = u / D$  it follows that  $w$  is maximum at

$$- D'(0)dt = 2a_{NG}(0)dt. \quad (4.5)$$

In all cases  $\bar{a} \neq \bar{0}$  replaces  $\bar{x}$  by  $\bar{x} - \bar{a}(\bar{0})dt$  in the exponent.

### 5. The Integral for the Random Paths

The (Riemannian) sum of consecutive increments converges in the mean square sense when in each time interval the most probable value is taken, together with the Gaussian deviations given by the second derivatives of the density at the top (these must be negative). Note that the increments (3.1) – as well as (2.2) – are Gaussian distributed, by the fact that all coefficients are fixed at their initial value  $\bar{x}$ . As the preceding Chapter 4 shows, (3.1) agrees with the oblique propagator (in the above way) when  $\alpha = 1$  for the FPE, and therefore  $\alpha = 0$  for the SDE. This singles out the Ito paths, and their increments are focused on the most probable values, determined by the FPE (3.3). This is the core of the paper, and it contrasts with the existing theory.

The oblique propagator provides some irrelevant information, like, e.g., its mean value. That phenomenon does not persist for possible steady densities solving (3.3), see also the Chapter 8. It is however essential for understanding the paradox that the exact (oblique) propagators do not satisfy the equation by Chapman-Kolmogorov CKE [10], while the resolving process is Markovian.



## 6. The Optimum Prediction

Given an initial  $\bar{x}_0$  it is interesting to know the most probable value of  $\bar{X}(t)$  at each  $t > 0$ . According to the general theory [1, 3], this is given by the conditional expectation, in view of (3.2) thus by the noiseless motion. The Ito-FPE would however yield a different value: already for small  $t$  the result (4.5) rather locates the maximum at  $\bar{x}_0 + [\bar{a}(\bar{x}_0) - 2\bar{a}_{Sp}(\bar{x}_0)]t$ . This is another flaw removed by the new FPE.

For a nonsingular  $\underline{D}$  one can say more about the density with the initial  $\delta(\bar{x} - \bar{x}_0)$ . After  $dt$  it is the Gaussian with mean  $\bar{x}_0 + \bar{a}(\bar{x}_0)dt$  and variance  $\underline{D}(\bar{x}_0)dt$ . Iterated use of the CKE with the Gaussian approximation of the propagator then shows that the maximum moves with the noiseless motion. The maximum property thus holds for each  $t > 0$ .

## 7. Change of the Variables

It is assumed that the SDE is consistent with any new variables  $\bar{y}(\bar{x})$ . In the noiseless case ( $\underline{B} = \underline{0}$ ,  $\dot{\bar{x}} = \bar{a}$ ) this means that  $\dot{y}^i = (dy^i / dx^k) \dot{x}^k = (dy^i / dx^k) a^k$ , so that  $\bar{a}$  is a contravariant vector. It is further supposed that  $\bar{a}$  does not depend on  $\underline{B} \neq \underline{0}$ . Since (by definition)  $\bar{x}$  is a contravariant vector, and  $\bar{W}(t)$  is understood to remain the same, the rows of  $\underline{B}$  are also contravariant vectors, and  $\underline{D}$  is a twice contravariant tensor, see also [5]. When  $\underline{D}$  is nonsingular, it can be transformed to unity by  $\bar{z}(\bar{x})$  given by  $d\bar{z} = [\underline{D}(\bar{x})]^{-1/2} d\bar{x}$ , i.e., by  $dz = b^{-1}(x)dx$  in one dimension, see [2, 4, 6]. Clearly  $\bar{a}_{Sp}$  then vanishes, which means that it is *not* a tensor. The conditional increment must thus not involve  $\bar{a}_{Sp}$ , whence  $\alpha = 0$  is the *only* possibility for the SDE. This restates the above finding in an independent way.

### 8. Steady Densities

It is now assumed that  $\bar{a}(\bar{x})$  excludes any escapes to infinity, so that a steady density  $w(\bar{x})$  may exist. It is a well-known idea to introduce a parameter  $\varepsilon$  indicating the noise strength. That  $\varepsilon$  then appears as a factor of  $\underline{D}$ ,  $\bar{a}_{NG}$  and  $\bar{a}_{Sp}$ . Consistently,  $w(\bar{x})$  is expressed as  $w(\bar{x}) := N(\varepsilon) \exp[-\phi(\bar{x})/\varepsilon]$  with the “quasipotential”  $\phi(\bar{x})$ , see, e.g., [11, 12]. It is then easily shown that (3.3) results in

$$\nabla\phi \cdot (\bar{a} + \underline{D}\nabla\phi/2) + \varepsilon\nabla \cdot (\bar{a} + \underline{D}\nabla\phi/2) = 0. \quad (8.1)$$

For weak noise one may neglect the second term. The remaining “eikonal equation”

$$\nabla\phi \cdot (\bar{a} + \underline{D}\nabla\phi/2) \approx 0 \quad (8.2)$$

holds in any variables  $\bar{y}(\bar{x})$ : The current (3.4) can be rewritten as  $\bar{J} = w(\bar{a} + \underline{D}\nabla\phi/2)$ , which shows that  $\bar{a} + \underline{D}\nabla\phi/2 := \bar{a}_c$  is a contravariant vector, and therefore  $\phi(\bar{x})$  a scalar. The scalar product  $\nabla\phi \cdot \bar{a}_c$  vanishes thus in all variables.

Stationarity for all  $\varepsilon > 0$  requires that

$$\nabla \cdot \bar{a}_c = 0, \quad (8.3)$$

and (8.2) is then exact. By (8.3)  $\bar{a}_c$  is source-free (solenoidal), a property that also holds for any  $\bar{y}(\bar{x})$ . It further follows that  $\phi(\bar{x})$  must be smooth.

“Detailed balance” [5] provides a solution satisfying (8.3). Since (8.2) is of the first order, it can be solved by characteristics [11]. In two dimensions a generating function  $\chi(\bar{x})$  obeys a simpler equation [12]. This becomes linear by (8.3), with the left-hand side  $\bar{a} \cdot \nabla\chi$ , and it can thus be solved in domains attracted by  $\bar{a}$  to a point.

Both the steady density and the increment (3.2) thus obey the tensor laws. This is new, and it strongly supports the new findings. The relevant equations (3.2) and (8.2) do not involve a derivative of  $\underline{D}$  and are thus the same, whether or not  $\underline{D}$  depends on  $\bar{x}$  (note that (4.4) was not used for the paths).

**Remark.** The above procedure is inappropriate for  $\alpha < 1$ : since  $\bar{a}_{Sp}$  is  $O(\varepsilon)$ , it would only appear in the second term of (8.1), but by (2.12)  $w(\bar{x})$  is maximum where  $\bar{a} + (\alpha - 1)\bar{a}_{Sp} = \bar{0}$ , while (8.2) states that  $\nabla\phi = \bar{0}$  where  $\bar{a} = \bar{0}$ .

Recall that the Ito paths follow the noiseless motion leading to the attractor of  $\bar{a}$ . There the steady density is maximum for the FPE (3.3) with  $\alpha = 1$ . This is not really new, but with the existing FPE it would not be consistent.

For advanced models with equilibrium states, see, e.g., [13].

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