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LIMITS OF SOLUTIONS OF A RECURRENCE RELATION WITH BANG BANG CONTROL

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Abstract

In this paper, we consider a three term nonlinear recurrence $x_n = ax_{n-2} + bH(x_{n-1}) + c$, where $0 < a < 1$, b, c are real numbers and H is the Heaviside step function. We are able to derive the exact relations between the initial values x_{-2} and x_{-1}, what's more, the limiting behaviours of the solution determined by them.

1. Introduction

Three term recurrence relations of the form

$$y_n = F(y_{n-1}, y_{n-2}), \quad n \in \mathbf{N} = \{0, 1, 2, 3, \ldots\},$$

arises in many studies of natural phenomena. A well known example has the relation

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$$y_n = y_{n-1} + y_{n-2}, \quad n \in \mathbf{N},$$

which is satisfied by the Fibonacci sequence $\{0, 1, 2, 3, 5, 8, \ldots\}$. Now, there are numorous studies for the above equation when F is a continuous function. However, when F is discontinuous, few studies are available (see, e.g., [1-4]). Since, (discontinuous) on-off control functions such as

$$H(u) = \begin{cases} 1 & \text{if } u \leq 0 \\ -1 & \text{if } u > 0 \end{cases}, \quad G(u) = \begin{cases} 1 & \text{if } u \geq 0 \\ -1 & \text{if } u < 0 \end{cases}, \tag{1}$$

or

$$H_\lambda(u) = \begin{cases} 1 & \text{if } u \leq \lambda \\ -1 & \text{if } u > \lambda \end{cases}, \lambda \in \mathbf{R}, \tag{2}$$

etc. are common, it is extremely important to consider prototype models and study their properties.

In this paper, we consider the following recurrence relation:

$$y_n = ay_{n-2} + bH_\lambda(y_{n-1}) + c, \quad n \in \mathbf{N}. \tag{3}$$

where $a \in (0, 1)$, $b, c \in \mathbf{R}$ and $H_\lambda : \mathbf{R} \to \mathbf{R}$ is the step (activation) or bang bang function defined by (2). Clearly, for given any initial pair (y_{-2}, y_{-1}) in $\mathbf{R}^2$, we can generate a unique real sequence $\{y_n\}_{n=-2}^{\infty}$ through (3). Such a sequence is called a solution of (3) originated from (y_{-2}, y_{-1}).

There are many qualitative properties of this nonlinear recurrence which are worthy of studying. Here, however, we will concentrate on one of its asymptotic behaviours. More specifically, given $(y_{-2}, y_{-1}) \in \mathbf{R}^2$, we are interested in the limit of the solution sequence $\{y_n\}_{n=-2}^{\infty}$ originated from it.

As we will see below, there are only a few types of limiting behaviours for solutions of (3) and we can also determine exactly the 'initial region' from which each type of solutions originate.

Since there are four real parameters in the nonlinear model (3), the above precise information may seem like a lot of trouble. Fortunately, we may resort to linear recurrences and transformations to solve this problem.

Indeed, let $\{y_k\}_{k=-2}^{\infty}$ be real sequences that satisfy

$$y_{2k} = ay_{2k-2} + d, \quad k \in \mathbf{N}, \tag{4}$$

$$y_{2k+1} = ay_{2k-1} + d, \quad k \in \mathbf{N}, \tag{5}$$

where $a \in (0, 1)$ and d is a real number. Then the following facts are obtained easily by induction:

- If $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (4), then

$$y_{2k} = a^{k+1}y_{-2} + \frac{(1 - a^{k+1})}{1 - a}d, \quad k \in \mathbf{N}. \tag{6}$$

- If $\{y_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (5), then

$$y_{2k+1} = a^{k+1}y_{-1} + \frac{(1 - a^{k+1})}{1 - a}d, \quad k \in \mathbf{N}. \tag{7}$$

Next, we assume that $b \neq 0$. Then by $x_n = y_n - \lambda$, (3) can be written as

$$x_n = ax_{n-2} + bH(x_{n-1}) + (c + (a - 1)\lambda), \quad n \in \mathbf{N}, \tag{8}$$

where H is the Heaviside function defined in (1). Furthermore, if $a \in (0, 1)$ and $b > 0$, then by $z_n = \frac{1 - a}{b}x_n$ and (8), we can get

$$z_n = az_{n-2} + (1 - a)H(z_{n-1}) + d, \quad n \in \mathbf{N}; \tag{9}$$

while if $a \in (0, 1)$ and $b < 0,$ then by $z_n = \frac{1-a}{b} x_n,$ (8) is equivalent to

$$z_n = az_{n-2} + (1-a)G(z_{n-1}) + d, \quad n \in \mathbf{N}, \tag{10}$$

where G is the Heaviside function defined in (1) and

$$d = \frac{1-a}{b}(c + (a-1)\lambda).$$

Since (10) is similar to (9), we consider the following equation:

$$x_n = ax_{n-2} + bH(x_{n-1}) + c, \quad n \in \mathbf{N}, \tag{11}$$

which includes (9) by assuming the case $a \in (0, 1),\ b > 0.$

Henceforth, we will discuss the limiting behaviours of solutions of (11).

To state the corresponding results, it is convenient to introduce some notations. We set

$$\alpha_+ = \frac{c \pm b}{1-a},\ a_k^{\pm} = \frac{1}{a^k}\left(-\frac{1-a^k}{1-a}(c \pm b)\right), \quad k \in \mathbf{N},$$

Next, if I and J are real intervals, their cross product $I \times J$ will be denoted by IJ, and we will assume that this product receives the priority attention in a mathematical expression. For instance, if we set

$$\mathbf{R}^- = (-\infty, 0], \quad \mathbf{R}^+ = (0, \infty),$$

then $\{\mathbf{R}^+\mathbf{R}^+, \mathbf{R}^+\mathbf{R}^-, \mathbf{R}^-\mathbf{R}^+, \mathbf{R}^-\mathbf{R}^-\}$ is a partition of $\mathbf{R}^2.$ Other subsets of the plane will be introduced in the subsequent sections. Here we will employ the following notations:

$$a + \Omega = \{a + x | x \in \Omega\},\ a\Omega = \{ax|\ x \in \Omega\}, \text{ for any } \Omega \subseteq \mathbf{R} \text{ and } a \in \mathbf{R}.$$

2. The Main Result

Under the assumption that $a \in (0, 1)$ we have $\alpha_- = (c - b)/(1 - a) < (c + b)/(1 - a) = \alpha_+$. Thus we need to consider five cases (i) $0 < \alpha_-$, (ii) $0 = \alpha_-$, (iii) $\alpha_- < 0 < \alpha_+$, (iv) $0 = \alpha_+$, and (v) $0 > \alpha_+$.

Theorem 2.1. *Suppose* $a \in (0, 1)$, $b > 0$ *and* $\alpha_+ < 0$. *Then every solution of* (11) *tends to* α_+.

Proof. Let $\{x_k\}_{k=-2}^{\infty}$ be a solution of (11). If $x_k > 0$ for all $k \geq -2$, then from (11), $x_k = ax_{k-2} - b + c$ for all $k \in \mathbf{N}$. One sees immediately from (6) and (7) that $\lim_n x_n = (c - b)/(1 - a) = \alpha_- < \alpha_+ < 0$, which is a contradiction. Thus there is $m_0 \geq -2$ such that $x_{m_0} \in \mathbf{R}^-$. Then we may show that there exists $m \geq -2$ such that $x_m, x_{m+1} \in \mathbf{R}^-$. In fact, if $x_{m_0+1} \in \mathbf{R}^-$, then we may take $m = m_0$, if $x_{m_0+1} \in \mathbf{R}^+$, then by (11), we have

$$x_{m_0+2} = ax_{m_0} + bH_\lambda(x_{m_0+1}) + c = ax_{m_0} - b + c \in \mathbf{R}^-,$$

$$x_{m_0+3} = ax_{m_0+1} + bH_\lambda(x_{m_0+2}) + c = ax_{m_0+1} + b + c \in \mathbf{R}^+,$$

$$\cdots\cdots$$

$$x_{m_0+2k} = ax_{m_0+2k-2} + bH_\lambda(x_{m_0+2k-1}) + c = ax_{m_0+2k-2} - b + c \in \mathbf{R}^-,$$

$$x_{m_0+2k+1} = ax_{m_0+2k-1} + bH_\lambda(x_{m_0+2k}) + c = ax_{m_0+2k-1} + b + c \in \mathbf{R}^+.$$

Thus,

$$\lim_{k\to\infty} x_{m_0+2k} = \alpha_- < 0, \quad \lim_{k\to\infty} x_{m_0+2k+1} = \alpha_+ < 0.$$

Therefore, there exists $m \geq -2$ such that $x_m, x_{m+1} \in \mathbf{R}^-$. We may suppose without loss of generality that $x_{-2}, x_{-1} \in \mathbf{R}^-$. Then by (11) and induction, $x_n \in \mathbf{R}^-$ for all $n \geq -2$. Thus $x_n = ax_{n-2} + b + c$ for $n \in \mathbf{N}$. In view of (6) and (7), $\lim_{n\to\infty} x_n = \alpha_+$. The proof is complete.

Theorem 2.2. *Suppose* $a \in (0, 1)$, $b > 0$ *and* $\alpha_- > 0$. *Then every solution of* (11) *tends to* α_-.

The proof is similar to that of Theorem 2.1 and hence is omitted. In the next result, we assume that $a \in (0, 1)$, $b > 0$ and $\alpha_+ = 0$. Then $0 = a_0^- < a_1^- < \cdots < a_k^- \to +\infty$. If we let

$$A^{(k)} = (a_k^-, a_{k+1}^-], \quad k \in \mathbf{N}, \tag{12}$$

and

$$A^{(-1)} = aA^{(0)} - b + c = (-b + c, 0],$$

then

$$A^{(k-1)} = aA^{(k)} - b + c, \; k \in \mathbf{N}, \; A^{(-1)} \subseteq \mathbf{R}^-, \; \mathbf{R}^+ = \bigcup_{k=0}^{\infty} A^{(k)}.$$

Theorem 2.3. *Suppose* $a \in (0, 1)$, $b > 0$ *and* $0 = \alpha_+$. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be any solution of* (11).

(i) *If* $(x_{-2}, x_{-1}) \in \mathbf{R}^-\mathbf{R}^-$, *then* $\lim_n x_n = 0$.

(ii) *If* $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup \mathbf{R}^+\mathbf{R}^-$, *where* $0 \le s < k$, *then* $\lim_n x_{2n} = 0$ *and* $\lim_n x_{2n+1} = \alpha_-$.

(iii) *If* $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup \mathbf{R}^-\mathbf{R}^+$, *where* $0 \le k \le s$, *then* $\lim_n x_{2n} = \alpha_-$ *and* $\lim_n x_{2n+1} = 0$.

Proof. The proof of (i) is quite easy and hence skipped. To see (ii), suppose $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)}$, where $0 \le s < k$. Then by (11),

$$x_0 = ax_{-2} + bH(x_{-1}) + c = ax_{-2} - b + c \in aA^{(k)} - b + c = A^{(k-1)},$$

$$x_1 = ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \in aA^{(s)} - b + c = A^{(s-1)},$$

and by induction, $(x_{2s}, x_{2s+1}) \in A^{(k-s-1)}A^{(-1)} \subseteq \mathbf{R}^+\mathbf{R}^-$. Therefore, we may suppose without loss of generality $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then by (11) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^+\mathbf{R}^-$ for all $n \geq -2$. Thus $x_{2n} = ax_{2n-2} + b + c$ and $x_{2n+1} = ax_{2n-1} - b + c$. In view of (6) and (7), $\lim_{n\to\infty} x_{2n} = 0$ and $\lim_{n\to\infty} x_{2n+1} = \alpha_-$ as desired. Finally, the proof of (iii) is similar to that of the case (ii) and hence omitted.

In the next result, we assume that $a \in (0, 1)$, $b > 0$ and $\alpha_- = 0$. Then $0 = a_0^+ > a_1^+ > \cdots > a_k^+ \to -\infty$ and $(-\infty, 0) = \bigcup_{k=0}^{\infty}[a_{k+1}^+, a_k^+)$.

In the next result, we assume that $a \in (0, 1)$, $b > 0$ and $\alpha_- < 0 < \alpha_+$. Then $0 = a_0^- < a_1^- < \cdots < a_k^- \to +\infty$ and $0 = a_0^+ > a_1^+ > \cdots > a_k^+ \to -\infty$. Therefore, if let $A^{(k)} = (a_k^-, a_{k+1}^-]$ and $B^{(k)} = (a_{k+1}^+, a_k^+]$ for $k \in \mathbf{N}$ and $A^{(-1)} = aA^{(0)} - b + c$ and $B^{(-1)} = aB^{(0)} + b + c$, then

$$aA^{(k)} - b + c = A^{(k-1)}, \quad aB^{(k)} + b + c = B^{(k-1)}, \quad k \in \mathbf{N},$$

$$A^{(-1)} \subseteq \mathbf{R}^-,\ B^{(-1)} \subseteq \mathbf{R}^+,\ \mathbf{R}^+ = \bigcup_{k=0}^{\infty} A^{(k)},\ \mathbf{R}^- = \bigcup_{k=0}^{\infty} B^{(k)}.$$

Theorem 2.4. *Suppose* $a \in (0, 1)$, $b > 0$ *and* $\alpha_- < 0 < \alpha_+$. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be any solution of* (11).

(i) *If* $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup B^{(r)}B^{(t)} \cup \mathbf{R}^+\mathbf{R}^-$, *where* $0 \leq s < k$ *and* $0 \leq r \leq t$, *then* $\lim_n x_{2n} = \alpha_+$ *and* $\lim_n x_{2n+1} = \alpha_-$.

(ii) *If* $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)} \cup B^{(r)}B^{(t)} \cup \mathbf{R}^-\mathbf{R}^+$, *where* $0 \leq k \leq s$ *and* $0 \leq t \leq r$, *then* $\lim_n x_{2n} = \alpha_-$ *and* $\lim_n x_{2n+1} = \alpha_+$.

Proof. Suppose $(x_{-2}, x_{-1}) \in A^{(k)}A^{(s)}$, where $0 \leq s < k$. Then by (11),

$$x_0 = ax_{-2} + bH(x_{-1}) + c = ax_{-2} - b + c \in aA^{(k)} - b + c = A^{(k-1)},$$

$$x_1 = ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \in aA^{(s)} - b + c = A^{(s-1)},$$

and by induction, $(x_{2s}, x_{2s+1}) \in A^{(k-s-1)}A^{(-1)} \subseteq \mathbf{R}^+\mathbf{R}^-$. Suppose $(x_{-2}, x_{-1}) \in B^{(r)}B^{(t)}$, where $0 \leq r \leq t$. By (11), if $r = 0$, then

$$x_0 = ax_{-2} + bH(x_{-1}) + c = ax_{-2} + b + c \in aB^{(0)} + b + c = B^{(-1)} \subseteq \mathbf{R}^+,$$

$$x_1 = ax_{-1} + bH(x_0) + c = ax_{-1} - b + c \leq -b + c < 0,$$

i.e., $(x_0, x_1) \in \mathbf{R}^+\mathbf{R}^-$; while if $r > 0$, then

$$x_0 = ax_{-2} + bH(x_{-1}) + c = ax_{-2} + b + c \in aB^{(r)} + b + c = B^{(r-1)},$$

$$x_1 = ax_{-1} + bH(x_0) + c = ax_{-1} + b + c \in aB^{(t)} + b + c = B^{(t-1)},$$

and by induction, $(x_{2r}, x_{2r+1}) \in \mathbf{R}^+\mathbf{R}^-$. Therefore, we may suppose without loss of generality that $(x_{-2}, x_{-1}) \in \mathbf{R}^+\mathbf{R}^-$. Then by (11) and induction, $(x_{2n}, x_{2n+1}) \in \mathbf{R}^+\mathbf{R}^-$ for all $n \geq -2$. Thus $x_{2n} = ax_{2n-2} + b + c$ and $x_{2n+1} = ax_{2n-1} - b + c$ for $n \in \mathbf{N}$. In view of (6) and (7), $\lim_{n\to\infty} x_{2n} = \alpha_+$ and $\lim_{n\to\infty} x_{2n+1} = \alpha_-$ as desired. The conclusion (ii) is similar to (i) and its proof is omitted.

Theorem 2.5. *Suppose* $\alpha \in (0, 1)$, $b > 0$ *and* $\alpha_- = 0$. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be any solution of* (11). *Then its limiting behaviour can be summarized in the following table*:

x_{-2}	x_{-1}	Condition	x_{2n}	x_{2n+1}
$\in \mathbf{R}^+$	$\in \mathbf{R}^+$		$\to 0$	$\to 0$
$\in (a_1^+, +\infty)$	$= 0$		$\to \alpha_+$	$\to 0$
$\in \mathbf{R}^-$	$\in \mathbf{R}^+$		$\to 0$	$\to \alpha_+$
$\in (-\infty, a_1^+]$	$= 0$		$\to 0$	$\to \alpha_+$
$\in [0, +\infty)$	$\in (-\infty, 0)$		$\to \alpha_+$	$\to 0$
$\in [a_{k+1}^+, a_k^+)$	$\in [a_{s+1}^+, a_s^+)$	$0 \le k < s$	$\to \alpha_+$	$\to 0$
$\in [a_{k+1}^+, a_k^+)$	$= a_{s+1}^+$	$0 \le k = s$	$\to \alpha_+$	$\to 0$
$= a_{k+1}^+$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s$	$\to 0$	$\to \alpha_+$
$\in (a_{k+1}^+, a_k^+)$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s$	$\to \alpha_+$	$\to 0$
$= a_{k+1}^+$	$\in [a_{s+1}^+, a_s^+)$	$0 \le k = s + 1$	$\to 0$	$\to \alpha_+$
$\in (a_{k+1}^+, a_k^+)$	$= a_{s+1}^+$	$0 \le k = s + 1$	$\to \alpha_+$	$\to 0$
$\in (a_{k+1}^+, a_k^+)$	$\in (a_{s+1}^+, a_s^+)$	$0 \le k = s + 1$	$\to 0$	$\to \alpha_+$
$\in [a_{k+1}^+, a_k^+)$	$\in [a_{s+1}^+, a_s^+)$	$k > s + 1$	$\to 0$	$\to \alpha_+$

The proof is similar to that of Theorems 2.1, 2.3, 2.4 and hence is omitted.

3. Concluding Remarks

Since we have derived the exact relations between the initial pair (x_{-2}, x_{-1}) with the limiting behaviours of the solution $\{x_k\}_{k=-2}^{\infty}$ of (11) originated from it, we may make some interesting observations. A solution $\{x_k\}_{k=-2}^{\infty}$ of (11) converges if, and only if, (i) $\alpha_+ < 0$, (ii) $\alpha_- > 0$, (iii) $\alpha_+ = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^-$; or, (iv) $\alpha_- = 0$ and $x_{-2}, x_{-1} \in \mathbf{R}^+$.

We may also make assertions on the limiting behaviours of subsequences $\{x_{2k}\}_{k=-1}^{\infty}$ and $\{x_{2k+1}\}_{k=-1}^{\infty}$ of solutions $\{x_k\}_{k=-2}^{\infty}$ of (11). These and others can be made by going through the previous results one by one, and are not listed here since they do not offer any new theoretical information.

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