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# GAUGE TRANSFORMATION, MOMENTS AND GENERATING FUNCTIONS FOR HIGHER-ORDER LAGRANGIANS

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## Abstract

This paper investigates the relations which appear between two higher-order Lagrange functions joined by a transformation of gauge type. Also, in the second part of this work, the change of variables in Hamiltonian and the generating function via higher-order Lagrangians are studied.

## 1. Introduction

The classical Lagrangian dynamics is governed by second order ordinary differential equations or second order partial differential equations (the multi-time case) of Euler-Lagrange type with boundary conditions. The Euler-Lagrange ODEs (PDEs) solutions are called

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*extremals* or *critical points* of the functionals considered (simple, multiple, curvilinear integrals). The integrating functions, named *Lagrange functions* or *Lagrangians*, are differentiable functions with vector argument. On the other hand, the classical Hamiltonian dynamics is formulated using first order ordinary differential equations or first order partial differential equations (the multi-time case) arising from second order Euler-Lagrange ODEs (PDEs). This transition is made using the *Legendre transformation*. For more details regarding the multi-time Lagrange, Hamilton and Hamilton-Jacobi dynamics, the reader is directed to Udrişte and Ţevy [11], [12], Treanţă [8], [10], Motta and Rampazzo [4], and Rochet [5].

The present work represents a natural continuation of a recent paper (Treanţă [8]), where only the case of second-order Lagrangians is considered. Thus, we extend, unify and improve several results in the current literature. Other different but connected ideas to this subject can be read in Miron [3], Krupkova [2], Roman [6], Ibragimov [1], Treanţă and Vârsan [9].

## 2. Gauge Transformation and Moments Governed by Higher-Order Lagrangians

In this section, we shall study the relations which appear between two Lagrange functions joined by a *gauge transformation*.

*The single-time case*. Let us consider two single-time higher-order Lagrangians,

$$L^{\zeta}\left(t,\, x(t),\, x^{(1)}(t),\, x^{(2)}(t),\, \ldots,\, x^{(k)}(t)\right), \quad \zeta = 1,\, 2,$$

with $t \in [t_0,\, t_1] \subset R$, $x(\cdot) \in R^n$, $k \geq 2$ a fixed natural number, joined by a transformation of *gauge* type (adding a total derivative), i.e.,

$$\begin{aligned} L^2 &= L^1 + \frac{d}{dt} f\left(t,\, x(t),\, x^{(1)}(t),\, x^{(2)}(t),\, \ldots,\, x^{(k-1)}(t)\right) \\ &= L^1 + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^j} x^{(1)j} + \frac{\partial f}{\partial x^{(1)j}} x^{(2)j} + \ldots + \frac{\partial f}{\partial x^{(k-1)j}} x^{(k)j}, \end{aligned}$$

$$j = \overline{1,\, n}.$$

The summation over the repeated indices is assumed. Then, the corresponding moments $p^1_{ai}$, $p^2_{ai}$, $a = \overline{1, k}$, $i = \overline{1, n}$, satisfy the following relations:

$$p^2_{ai} := \frac{\partial L^2}{\partial x^{(a)i}} = \frac{\partial L^1}{\partial x^{(a)i}} + \frac{d}{dt}\frac{\partial f}{\partial x^{(a)i}} + \frac{\partial f}{\partial x^{(a-1)i}}$$

$$= p^1_{ai} + \frac{d}{dt}\frac{\partial f}{\partial x^{(a)i}} + \frac{\partial f}{\partial x^{(a-1)i}}, \quad a = \overline{1, k-1},$$

$$p^2_{ki} := \frac{\partial L^2}{\partial x^{(k)i}} = \frac{\partial L^1}{\partial x^{(k)i}} + \frac{\partial f}{\partial x^{(k-1)i}} = p^1_{ki} + \frac{\partial f}{\partial x^{(k-1)i}}, \quad a = k.$$

The previous computations allow us to establish the following result.

**Proposition 2.1.** *Two single-time higher-order Lagrangians, satisfying* $L^2 = L^1 + \frac{d}{dt} f\left(t, x(t), x^{(1)}(t), x^{(2)}(t), \ldots, x^{(k-1)}(t)\right)$, *where* $L^2$, $L^1$, *and f are considered* $C^{k+1}$*-class functions, produce the same Euler-Lagrange ODEs, i.e.,*

$$\frac{\partial L^2}{\partial x^i} - \frac{d}{dt}\frac{\partial L^2}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial L^2}{\partial x^{(2)i}} - \ldots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L^2}{\partial x^{(k)i}}$$

$$= \frac{\partial L^1}{\partial x^i} - \frac{d}{dt}\frac{\partial L^1}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial L^1}{\partial x^{(2)i}} - \ldots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L^1}{\partial x^{(k)i}}, \quad i = \overline{1, n},$$

*or*

$$\sum_{r=1}^{k+1}(-1)^{r-1}\frac{d^{r-1}}{dt^{r-1}}\frac{\partial L^2}{\partial x^{(r-1)i}} = \sum_{r=1}^{k+1}(-1)^{r-1}\frac{d^{r-1}}{dt^{r-1}}\frac{\partial L^1}{\partial x^{(r-1)i}}, \quad i = \overline{1, n}.$$

**Proof.** By direct computation, we get

$$\frac{\partial L^2}{\partial x^i} - \frac{d}{dt}\frac{\partial L^2}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial L^2}{\partial x^{(2)i}} - \ldots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L^2}{\partial x^{(k)i}}$$

$$= \sum_{r=1}^{k}(-1)^{r-1}\frac{d^{r-1}}{dt^{r-1}}\frac{\partial L^2}{\partial x^{(r-1)i}} + (-1)^k \frac{d^k}{dt^k}\frac{\partial L^2}{\partial x^{(k)i}}$$

$$= \frac{\partial L^2}{\partial x^i} + \sum_{r=2}^{k} (-1)^{r-1} \frac{d^{r-1}}{dt^{r-1}} \left( \frac{\partial L^1}{\partial x^{(r-1)i}} + \frac{d}{dt} \frac{\partial f}{\partial x^{(r-1)i}} + \frac{\partial f}{\partial x^{(r-2)i}} \right)$$

$$+ (-1)^k \frac{d^k}{dt^k} \left( \frac{\partial L^1}{\partial x^{(k)i}} + \frac{\partial f}{\partial x^{(k-1)i}} \right)$$

$$= \frac{\partial L^1}{\partial x^i} + \frac{d}{dt} \frac{\partial f}{\partial x^i} + \sum_{r=2}^{k} (-1)^{r-1} \frac{d^{r-1}}{dt^{r-1}} \frac{\partial L^1}{\partial x^{(r-1)i}} + (-1)^k \frac{d^k}{dt^k} \frac{\partial L^1}{\partial x^{(k)i}}$$

$$+ (-1)^k \frac{d^k}{dt^k} \frac{\partial f}{\partial x^{(k-1)i}} - \frac{d}{dt} \frac{\partial f}{\partial x^i} + (-1)^{k-1} \frac{d^k}{dt^k} \frac{\partial f}{\partial x^{(k-1)i}}$$

$$= \sum_{r=1}^{k} (-1)^{r-1} \frac{d^{r-1}}{dt^{r-1}} \frac{\partial L^1}{\partial x^{(r-1)i}} + (-1)^k \frac{d^k}{dt^k} \frac{\partial L^1}{\partial x^{(k)i}}$$

$$= \frac{\partial L^1}{\partial x^i} - \frac{d}{dt} \frac{\partial L^1}{\partial x^{(1)i}} + \frac{d^2}{dt^2} \frac{\partial L^1}{\partial x^{(2)i}} - \dots + (-1)^k \frac{d^k}{dt^k} \frac{\partial L^1}{\partial x^{(k)i}}, \quad i = \overline{1, n},$$

and the proof is complete.

*The multi-time case.* Consider two multi-time higher-order Lagrangians,

$$L^{\zeta}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_k}(t)), \quad \zeta = 1, 2,$$

that are joined by a transformation of gauge type (adding a total derivative), i.e.,

$$L^2 = L^1 + D_\eta f^\eta(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t))$$

$$= L^1 + \frac{\partial f^\eta}{\partial t^\eta} + \frac{\partial f^\eta}{\partial x^j} x^j_\eta + \frac{\partial f^\eta}{\partial x^j_{\alpha_1}} x^j_{\alpha_1 \eta} + \frac{1}{n(\alpha_1, \alpha_2)} \frac{\partial f^\eta}{\partial x^j_{\alpha_1 \alpha_2}} x^j_{\alpha_1 \alpha_2 \eta}$$

$$+ \dots + \frac{1}{n(\alpha_1, \dots, \alpha_{k-1})} \frac{\partial f^\eta}{\partial x^j_{\alpha_1 \dots \alpha_{k-1}}} x^j_{\alpha_1 \dots \alpha_{k-1} \eta}, \quad \eta = \overline{1, m}, \quad j = \overline{1, n}. \tag{2.1}$$

Here $t = (t^1, \dots, t^m) \in \Omega_{t_0, t_1} \subset R^m$ (see $\Omega_{t_0, t_1}$ as the hyper-parallelepiped determined by diagonal opposite points $t_0, t_1$ from $R^m$), $x_{\alpha_1}(t) := \frac{\partial x}{\partial t^{\alpha_1}}(t), \dots, x_{\alpha_1 \dots \alpha_k}(t) := \frac{\partial^k x}{\partial t^{\alpha_1} \dots \partial t^{\alpha_k}}(t)$, $\alpha_j \in \{1, 2, \dots, m\}$, $j = \overline{1, k}$, $x : \Omega_{t_0, t_1} \subset R^m \to R^n$, $x = (x^i)$, $i \in \{1, 2, \dots, n\}$, and $n(\alpha_1, \alpha_2, \dots, \alpha_k) = \frac{|1_{\alpha_1} + 1_{\alpha_2} + \dots + 1_{\alpha_k}|!}{(1_{\alpha_1} + 1_{\alpha_2} + \dots + 1_{\alpha_k})!}$ (for more detail, see Saunders [7] and Treanţă [8]). The summation over the repeated indices is assumed.

The corresponding moments $p_{i1}^{\alpha_1 \dots \alpha_j}$, $p_{i2}^{\alpha_1 \dots \alpha_j}$, $j = \overline{1, k}$, $i = \overline{1, n}$, satisfy the following relations:

$$p_{i2}^{\alpha_1 \dots \alpha_j} := \frac{\partial L^2}{\partial x^i_{\alpha_1 \dots \alpha_j}} = \frac{\partial L^1}{\partial x^i_{\alpha_1 \dots \alpha_j}} + \frac{\partial}{\partial x^i_{\alpha_1 \dots \alpha_j}}(D_\eta f^\eta)$$

$$= p_{i1}^{\alpha_1 \dots \alpha_j} + D_\eta \frac{\partial f^\eta}{\partial x^i_{\alpha_1 \dots \alpha_j}} + \frac{1}{n(\alpha_1, \dots, \alpha_{j-1})} \frac{\partial f^{\alpha_j}}{\partial x^i_{\alpha_1 \dots \alpha_{j-1}}}, \quad j = \overline{1, k-1},$$

$$p_{i2}^{\alpha_1 \dots \alpha_k} := \frac{\partial L^2}{\partial x^i_{\alpha_1 \dots \alpha_k}} = \frac{\partial L^1}{\partial x^i_{\alpha_1 \dots \alpha_k}} + \frac{1}{n(\alpha_1, \dots, \alpha_{k-1})} \frac{\partial f^{\alpha_k}}{\partial x^i_{\alpha_1 \dots \alpha_{k-1}}}$$

$$= p_{i1}^{\alpha_1 \dots \alpha_k} + \frac{1}{n(\alpha_1, \dots, \alpha_{k-1})} \frac{\partial f^{\alpha_k}}{\partial x^i_{\alpha_1 \dots \alpha_{k-1}}}, \quad j = k.$$

Taking into account the previous computations, we establish the following result.

**Proposition 2.2.** *Two multi-time higher-order Lagrangians, satisfying* $L^2 = L^1 + D_\eta f^\eta(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t))$, *where* $L^2$, $L^1$, *and f are considered* $C^{k+1}$*-class functions, produce the same PDEs, i.e.,*

$$\frac{\partial L^2}{\partial x^i} - D_{\alpha_1}\frac{\partial L^2}{\partial x^i_{\alpha_1}} + D^2_{\alpha_1\alpha_2}\frac{\partial L^2}{\partial x^i_{\alpha_1\alpha_2}} - D^3_{\alpha_1\alpha_2\alpha_3}\frac{\partial L^2}{\partial x^i_{\alpha_1\alpha_2\alpha_3}}$$

$$+ \dots + (-1)^k D^k_{\alpha_1\alpha_2\dots\alpha_k}\frac{\partial L^2}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_k}}$$

$$= \frac{\partial L^1}{\partial x^i} - D_{\alpha_1}\frac{\partial L^1}{\partial x^i_{\alpha_1}} + D^2_{\alpha_1\alpha_2}\frac{\partial L^1}{\partial x^i_{\alpha_1\alpha_2}} - D^3_{\alpha_1\alpha_2\alpha_3}\frac{\partial L^1}{\partial x^i_{\alpha_1\alpha_2\alpha_3}}$$

$$+ \dots + (-1)^k D^k_{\alpha_1\alpha_2\dots\alpha_k}\frac{\partial L^1}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_k}}, \quad i \in \{1, 2, \dots, n\},$$

*or, shortly,*

$$\sum_{r=0}^{k}(-1)^r D^r_{\alpha_1\alpha_2\dots\alpha_r}\frac{\partial L^2}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_r}} = \sum_{r=0}^{k}(-1)^r D^r_{\alpha_1\alpha_2\dots\alpha_r}\frac{\partial L^1}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_r}}, \quad i = \overline{1, n},$$

*if and only if the total divergence of f is defined as*

$$D_\eta f^\eta(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{k-1}}(t))$$

$$= \frac{\partial f^\eta}{\partial t^\eta} + \frac{\partial f^\eta}{\partial x^j}x^j_\eta + \frac{\partial f^\eta}{\partial x^j_{\alpha_1}}x^j_{\alpha_1\eta} + \frac{\partial f^\eta}{\partial x^j_{\alpha_1\alpha_2}}x^j_{\alpha_1\alpha_2\eta}$$

$$+ \dots + \frac{\partial f^\eta}{\partial x^j_{\alpha_1\dots\alpha_{k-1}}}x^j_{\alpha_1\dots\alpha_{k-1}\eta}, \quad \eta = \overline{1, m}, \quad j = \overline{1, n}.$$

*Otherwise*, (*i.e.*, *if the total divergence of f is defined as in* (2.1)) *the following equality is true* (*i.e.*, $L^2$ *and* $L^1$ *produce the same PDEs*):

$$\sum_{r=0}^{k}(-1)^r D^r_{\alpha_1\dots\alpha_r}\frac{\partial L^2}{\partial x^i_{\alpha_1\dots\alpha_r}} = \sum_{r=0}^{k}(-1)^r D^r_{\alpha_1\dots\alpha_r}\frac{\partial L^1}{\partial x^i_{\alpha_1\dots\alpha_r}}, \quad i \in \{1, 2, \dots, n\},$$

*if and only if*

$$\sum_{r=2}^{k-1}(-1)^r D^{r+1}_{\alpha_1\ldots\alpha_r\eta}\frac{\partial f^\eta}{\partial x^i_{\alpha_1\ldots\alpha_r}}$$

$$= \sum_{r=3}^{k}(-1)^{r+1}\frac{1}{n(\alpha_1, \ldots, \alpha_{r-1})}D^r_{\alpha_1\ldots\alpha_{r-1}\alpha_r}\frac{\partial f^{\alpha_r}}{\partial x^i_{\alpha_1\ldots\alpha_{r-1}}},$$

$$i \in \{1, 2, \ldots, n\}.$$

**Proof.** Direct calculation.

**Corollary 2.1.** *Let us consider that the relation* (2.1) *is verified. Then,* $L^2$ *and* $L^1$ *produce the same multi-time Euler-Lagrange PDEs*, *i.e.*,

$$\sum_{r=0}^{k}(-1)^r\frac{1}{n(\alpha_1, \ldots, \alpha_r)}D^r_{\alpha_1\ldots\alpha_r}\frac{\partial L^2}{\partial x^i_{\alpha_1\ldots\alpha_r}}$$

$$= \sum_{r=0}^{k}(-1)^r\frac{1}{n(\alpha_1, \ldots, \alpha_r)}D^r_{\alpha_1\ldots\alpha_r}\frac{\partial L^1}{\partial x^i_{\alpha_1\ldots\alpha_r}}, \quad i \in \{1, 2, \ldots, n\},$$

*if and only if*

$$\sum_{r=1}^{k-1}(-1)^r\frac{1}{n(\alpha_1, \ldots, \alpha_r)}D^{r+1}_{\alpha_1\ldots\alpha_r\eta}\frac{\partial f^\eta}{\partial x^i_{\alpha_1\ldots\alpha_r}}$$

$$= \sum_{r=2}^{k}(-1)^{r+1}\frac{1}{n(\alpha_1, \ldots, \alpha_r)}\frac{1}{n(\alpha_1, \ldots, \alpha_{r-1})}D^r_{\alpha_1\ldots\alpha_{r-1}\alpha_r}\frac{\partial f^{\alpha_r}}{\partial x^i_{\alpha_1\ldots\alpha_{r-1}}},$$

$$i = \overline{1, n}.$$

**Remark 2.1.** The previous multi-time case takes into account the total divergence of $f$. As well, we can consider multi-time higher-order Lagrangian 1-forms, $L^\varepsilon_\zeta(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\ldots\alpha_k}(t))dt^\zeta$, $\varepsilon = 1, 2,$ and the transformation of *gauge* type becomes $L_\zeta^2 = L_\zeta^1 + D_\zeta f(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{k-1}}(t))$, $\zeta = \overline{1, m}$.

The corresponding moments $p_{i\zeta,1}^{\alpha_1\dots\alpha_j}$, $p_{i\zeta,2}^{\alpha_1\dots\alpha_j}$, $j = \overline{1, k}$, $i = \overline{1, n}$, satisfy the following relations:

$$p_{i\zeta,2}^{\alpha_1\dots\alpha_j} := \frac{\partial L_\zeta^2}{\partial x^i_{\alpha_1\dots\alpha_j}} = \frac{\partial L_\zeta^1}{\partial x^i_{\alpha_1\dots\alpha_j}} + \frac{\partial}{\partial x^i_{\alpha_1\dots\alpha_j}}(D_\zeta f)$$

$$= p_{i\zeta,1}^{\alpha_1\dots\alpha_j} + D_\zeta \frac{\partial f}{\partial x^i_{\alpha_1\dots\alpha_j}} + \frac{1}{n(\alpha_1, \dots, \alpha_p)} \frac{\partial f}{\partial x^i_{\alpha_1\dots\alpha_p}} \delta_\zeta^{\alpha_{p+1}\dots\alpha_j},$$

$$j = \overline{1, k-1},\; p = j - 1,$$

$$p_{i\zeta,2}^{\alpha_1\dots\alpha_k} := \frac{\partial L_\zeta^2}{\partial x^i_{\alpha_1\dots\alpha_k}} = \frac{\partial L_\zeta^1}{\partial x^i_{\alpha_1\dots\alpha_k}} + \frac{\partial}{\partial x^i_{\alpha_1\dots\alpha_k}}(D_\zeta f)$$

$$= p_{i\zeta,1}^{\alpha_1\dots\alpha_k} + \frac{1}{n(\alpha_1, \dots, \alpha_{k-1})} \frac{\partial f}{\partial x^i_{\alpha_1\dots\alpha_{k-1}}} \delta_\zeta^{\alpha_k}, \quad j = k.$$

Using the previous relations and $\frac{\partial L_\zeta^2}{\partial x^i} = \frac{\partial L_\zeta^1}{\partial x^i} + D_\zeta \frac{\partial f}{\partial x^i}$, we establish the following result.

**Proposition 2.3.** *Two multi-time higher-order Lagrangian* 1*-forms, satisfying* $L_\zeta^2 = L_\zeta^1 + D_\zeta f$, *where* $L_\zeta^2$, $L_\zeta^1$, *and* $f$ *are considered* $C^{k+1}$*-class functions, produce the same PDEs, i.e.*,

$$\sum_{r=0}^{k} (-1)^r D^r_{\alpha_1\alpha_2\dots\alpha_r} \frac{\partial L_\zeta^2}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_r}} = \sum_{r=0}^{k} (-1)^r D^r_{\alpha_1\alpha_2\dots\alpha_r} \frac{\partial L_\zeta^1}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_r}}, \quad i = \overline{1, n},$$

*if and only if*

$$\sum_{r=2}^{k-1}(-1)^r D^{r+1}_{\alpha_1\alpha_2\ldots\alpha_r\zeta} \frac{\partial f}{\partial x^i_{\alpha_1\alpha_2\ldots\alpha_r}}$$

$$= \sum_{r=3}^{k}(-1)^{r+1} \frac{1}{n(\alpha_1, \ldots, \alpha_p)} D^r_{\alpha_1\alpha_2\ldots\alpha_r} \frac{\partial f}{\partial x^i_{\alpha_1\alpha_2\ldots\alpha_p}} \delta_\zeta^{\alpha_{p+1}\ldots\alpha_r},$$

$$i = \overline{1, n}, \; p = r - 1.$$

**Proof.** Direct computation.

**Corollary 2.2.** *The multi-time higher-order Lagrangian* 1*-forms,* $L^2_\zeta$ *and* $L^1_\zeta$*, joined by a transformation of gauge type, produce the same multi-time Euler-Lagrange PDEs, i.e.,*

$$\sum_{r=0}^{k}(-1)^r \frac{1}{n(\alpha_1, \ldots, \alpha_r)} D^r_{\alpha_1\ldots\alpha_r} \frac{\partial L^2_\zeta}{\partial x^i_{\alpha_1\ldots\alpha_r}}$$

$$= \sum_{r=0}^{k}(-1)^r \frac{1}{n(\alpha_1, \ldots, \alpha_r)} D^r_{\alpha_1\ldots\alpha_r} \frac{\partial L^1_\zeta}{\partial x^i_{\alpha_1\ldots\alpha_r}}, \quad i \in \{1, 2, \ldots, n\},$$

*if and only if*

$$\sum_{r=1}^{k-1}(-1)^r \frac{1}{n(\alpha_1, \ldots, \alpha_r)} D^{r+1}_{\alpha_1\ldots\alpha_r\zeta} \frac{\partial f}{\partial x^i_{\alpha_1\ldots\alpha_r}}$$

$$= \sum_{r=2}^{k}(-1)^{r+1} \frac{1}{n(\alpha_1, \ldots, \alpha_r)} \frac{1}{n(\alpha_1, \ldots, \alpha_p)} D^r_{\alpha_1\ldots\alpha_r} \frac{\partial f}{\partial x^i_{\alpha_1\ldots\alpha_p}} \delta_\zeta^{\alpha_{p+1}\ldots\alpha_r},$$

$$i = \overline{1, n}, \; p = r - 1.$$

**2.1. The adding of the dissipative forces**

*The single-time case*. Let us consider the function

$$R\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k)}(t)\right),\qquad k\ \geq 1,$$

that determines the *generalized dissipative force* $-\dfrac{\partial R}{\partial x^{(k)i}}$. Such type of forces, for a fixed $k$, changes the ODEs of Hamiltonian type as

$$\sum_{a=1}^{k}(-1)^{a+1}\frac{d^{a}}{dt^{a}}\,p_{ai}=-\frac{\partial H}{\partial x^{i}}-\frac{\partial R}{\partial x^{(k)i}},\qquad \frac{d^{a}}{dt^{a}}\,x^{i}=\frac{\partial H}{\partial p_{ai}}.$$

We obtain

$$\frac{dH}{dt}=\frac{\partial H}{\partial x^{i}}\,x^{(1)i}+\frac{\partial H}{\partial p_{ai}}\frac{dp_{ai}}{dt}+\frac{\partial H}{\partial t}$$

$$=\left[\sum_{a=1}^{k}(-1)^{a}\frac{d^{a}}{dt^{a}}\,p_{ai}-\frac{\partial R}{\partial x^{(k)i}}\right]x^{(1)i}+x^{(a)i}\,\frac{dp_{ai}}{dt}+\frac{\partial H}{\partial t}$$

$$=\left(-\frac{\partial L}{\partial x^{i}}-\frac{\partial R}{\partial x^{(k)i}}\right)x^{(1)i}+x^{(a)i}\,\frac{dp_{ai}}{dt}+\frac{\partial H}{\partial t},$$

(summation over the repeated indices!) and $\dfrac{\partial H}{\partial t}=0$ implies

$$\frac{dH}{dt}=0\Leftrightarrow x^{(a)i}\,\frac{dp_{ai}}{dt}=\left(\frac{\partial L}{\partial x^{i}}+\frac{\partial R}{\partial x^{(k)i}}\right)x^{(1)i}.$$

*The multi-time case*. Let assume that the function

$$R(t,\ x(t),\ x_{\alpha_1}(t),\ \dots,\ x_{\alpha_1\dots\alpha_k}(t)),\qquad k\ \geq 1,$$

determines the *generalized multi-time* $\alpha_1\dots\alpha_k$*-dissipative force* $-\dfrac{\partial R}{\partial x^{i}_{\alpha_1\dots\alpha_k}}$, where $\alpha_1,\ \dots,\ \alpha_k\in\{1,\ 2,\ \dots,\ m\}$ are fixed. The PDEs of Hamiltonian type becomes

$$\sum_{j=1}^{k}(-1)^{j+1}D^{j}_{\alpha_1\ldots\alpha_j}p_i^{\alpha_1\ldots\alpha_j} = -\frac{\partial H}{\partial x^i} - \frac{\partial R}{\partial x^i_{\alpha_1\ldots\alpha_k}},$$

$$x^i_{\alpha_1\ldots\alpha_j} = \frac{\partial H}{\partial p_i^{\alpha_1\ldots\alpha_j}}, \quad j = \overline{1, k}, \quad i = \overline{1, n}.$$

Here, $p_i^{\alpha_1\ldots\alpha_j} := \dfrac{1}{n(\alpha_1, \ldots, \alpha_j)}\dfrac{\partial L}{\partial x^i_{\alpha_1\ldots\alpha_j}}$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, k\}$, $\alpha_j \in \{1, 2, \ldots, m\}$.

Computing the total derivative of $H$, we find

$$D_\zeta H = \frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\zeta} + \sum_{j=1}^{k}\frac{\partial H}{\partial p_i^{\alpha_1\ldots\alpha_j}}\frac{\partial p_i^{\alpha_1\ldots\alpha_j}}{\partial t^\zeta} + \frac{\partial H}{\partial t^\zeta}$$

$$= \left(\sum_{j=1}^{k}(-1)^{j}D^{j}_{\alpha_1\ldots\alpha_j}p_i^{\alpha_1\ldots\alpha_j} - \frac{\partial R}{\partial x^i_{\alpha_1\ldots\alpha_k}}\right)x^i_\zeta$$

$$+ \sum_{j=1}^{k}\frac{\partial H}{\partial p_i^{\alpha_1\ldots\alpha_j}}\frac{\partial p_i^{\alpha_1\ldots\alpha_j}}{\partial t^\zeta} + \frac{\partial H}{\partial t^\zeta},$$

and $\dfrac{\partial H}{\partial t^\zeta} = 0$ implies

$$D_\zeta H = \left(-\frac{\partial R}{\partial x^i_{\alpha_1\ldots\alpha_k}} - \frac{\partial L}{\partial x^i}\right)x^i_\zeta + \sum_{j=1}^{k}x^i_{\alpha_1\ldots\alpha_j}\frac{\partial p_i^{\alpha_1\ldots\alpha_j}}{\partial t^\zeta}.$$

## 3. The Change of Variables in Hamiltonian and the Generating Function via Higher-Order Lagrangians

*The single-time case.* Let $H = x^{(a)i} p_{ai} - L$ be the Hamiltonian and

$$\sum_{a=1}^{k} (-1)^{a+1} \frac{d^a}{dt^a} p_{ai}(t) = -\frac{\partial H}{\partial x^i}(x(t),\ p_1(t),\ \dots,\ p_k(t),\ t), \tag{3.1}$$

$$\frac{d^a}{dt^a} x^i(t) = \frac{\partial H}{\partial p_{ai}}(x(t),\ p_1(t),\ \dots,\ p_k(t),\ t), \quad i = \overline{1,\ n},$$

the associated ODEs. Let assume that we want to pass from our coordinates $(x^i,\ p_{1i},\ \dots,\ p_{ki},\ t)$ to the coordinates $(X^i,\ P_{1i},\ \dots,\ P_{ki},\ t)$ with the following change of variables (diffeomorphism):

$$X^{\eta} = X^{\eta}(x^i,\ p_{1i},\ \dots,\ p_{ki},\ t), \quad P_{1\eta} = P_{1\eta}(x^i,\ p_{1i},\ \dots,\ p_{ki},\ t),$$

$$\dots,\ P_{k\eta} = P_{k\eta}(x^i,\ p_{1i},\ \dots,\ p_{ki},\ t), \quad \eta \in \{1,\ 2,\ \dots,\ n\}.$$

Then, the Hamiltonian $H(x,\ p_1,\ \dots,\ p_k,\ t)$ changes in $K(X, P_1, \dots, P_k, t)$. The above change of variables is called *canonical transformation* if there is a Hamiltonian, $K(X,\ P_1,\ \dots,\ P_k,\ t)$, such that the associated ODEs

$$\sum_{a=1}^{k} (-1)^{a+1} \frac{d^a}{dt^a} p_{ai}(t) = -\frac{\partial K}{\partial X^i}(X(t),\ P_1(t),\ \dots,\ P_k(t),\ t),$$

$$\frac{d^a}{dt^a} X^i(t) = \frac{\partial K}{\partial P_{ai}}(X(t),\ P_1(t),\ \dots,\ P_k(t),\ t), \quad i = \overline{1,\ n},$$

and the ODEs (3.1) take place simultaneously. This thing is possible if the functions

$$x^{(a)i}(t) p_{ai}(t) - H(x(t),\ p_1(t),\ \dots,\ p_k(t),\ t),$$

and

$$X^{(a)i}(t)P_{ai}(t) - K(X(t), P_1(t), \ldots, P_k(t), t),$$

differ by a total derivative $\frac{dW}{dt}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)$.

**Lemma 3.1.** *If the Lagrangians*

$$L_1 := x^{(a)i}p_{ai} - H, \quad L_2 := X^{(a)i}P_{ai} - K,$$

*produce the same Euler-Lagrange ODEs, then the change of variables*

$$(x^i, p_{1i}, \ldots, p_{ki}, t) \hookrightarrow (X^i, P_{1i}, \ldots, P_{ki}, t), \quad i = \overline{1, n},$$

*is a canonical transformation.*

**Proof.** Using Proposition 2.1, the result is obvious.

The function $W$ is called the (*higher-order*) *generating function* of the canonical transformation.

*The multi-time case.* Let $H = x^i_{\alpha_1\ldots\alpha_j}p_i^{\alpha_1\ldots\alpha_j} - L$ be the Hamiltonian and

$$\sum_{j=1}^{k}(-1)^{j+1}D^j_{\alpha_1\ldots\alpha_j}p_i^{\alpha_1\ldots\alpha_j}(t) = -\frac{\partial H}{\partial x^i}(x(t), p^{\alpha_1}(t), \ldots, p^{\alpha_1\ldots\alpha_k}(t), t), \quad (3.2)$$

$$x^i_{\alpha_1\ldots\alpha_j}(t) = \frac{\partial H}{\partial p_i^{\alpha_1\ldots\alpha_j}}(x(t), p^{\alpha_1}(t), \ldots, p^{\alpha_1\ldots\alpha_k}(t), t), \quad i = \overline{1, n},$$

the associated PDEs. The summation over the repeated indices is assumed. Let assume that we want to pass from our coordinates $(x^i, p_i^{\alpha_1}, \ldots, p_i^{\alpha_1\ldots\alpha_k}, t)$ to the coordinates $(X^i, P_i^{\alpha_1}, \ldots, P_i^{\alpha_1\ldots\alpha_k}, t)$ with the following change of variables (diffeomorphism):

$$X^\eta = X^\eta(x^i, p_i^{\alpha_1}, \ldots, p_i^{\alpha_1\ldots\alpha_k}, t), \quad P_\eta^{\alpha_1} = P_\eta^{\alpha_1}(x^i, p_i^{\alpha_1}, \ldots, p_i^{\alpha_1\ldots\alpha_k}, t),$$

$$\ldots, P_\eta^{\alpha_1\ldots\alpha_k} = P_\eta^{\alpha_1\ldots\alpha_k}(x^i, p_i^{\alpha_1}, \ldots, p_i^{\alpha_1\ldots\alpha_k}, t), \quad \eta \in \{1, 2, \ldots, n\}.$$

Then, the Hamiltonian $H(x, p^{\alpha_1}, \ldots, p^{\alpha_1 \ldots \alpha_k}, t)$ changes in

$$K(X, P^{\alpha_1}, \ldots, P^{\alpha_1 \ldots \alpha_k}, t).$$

The above change of variables is called *canonical transformation* if there is a Hamiltonian, $K(X, P^{\alpha_1}, \ldots, P^{\alpha_1 \ldots \alpha_k}, t)$, such that the associated PDEs

$$\sum_{j=1}^{k} (-1)^{j+1} D^j_{\alpha_1 \ldots \alpha_j} p_i^{\alpha_1 \ldots \alpha_j}(t) = -\frac{\partial K}{\partial X^i}(X(t), P^{\alpha_1}(t), \ldots, P^{\alpha_1 \ldots \alpha_k}(t), t),$$

$$X^i_{\alpha_1 \ldots \alpha_j}(t) = \frac{\partial K}{\partial P_i^{\alpha_1 \ldots \alpha_j}}(X(t), P^{\alpha_1}(t), \ldots, P^{\alpha_1 \ldots \alpha_k}(t), t), \quad i = \overline{1, n},$$

and the PDEs (3.2) take place simultaneously. This thing is possible if the functions

$$x^i_{\alpha_1 \ldots \alpha_j}(t) p_i^{\alpha_1 \ldots \alpha_j}(t) - H(x(t), p^{\alpha_1}(t), \ldots, p^{\alpha_1 \ldots \alpha_k}(t), t),$$

and

$$X^i_{\alpha_1 \ldots \alpha_j}(t) P_i^{\alpha_1 \ldots \alpha_j}(t) - K(X(t), P^{\alpha_1}(t), \ldots, P^{\alpha_1 \ldots \alpha_k}(t), t),$$

differ by a total divergence $D_\zeta W^\zeta(x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1 \ldots \alpha_{k-1}}(t), t)$, $\zeta = \overline{1, m}$, and

$$\sum_{r=1}^{k-1} (-1)^r \frac{1}{n(\alpha_1, \ldots, \alpha_r)} D^{r+1}_{\alpha_1 \ldots \alpha_r \zeta} \frac{\partial W^\zeta}{\partial x^i_{\alpha_1 \ldots \alpha_r}}$$

$$= \sum_{r=2}^{k} (-1)^{r+1} \frac{1}{n(\alpha_1, \ldots, \alpha_r)} \frac{1}{n(\alpha_1, \ldots, \alpha_{r-1})} D^r_{\alpha_1 \ldots \alpha_{r-1} \alpha_r} \frac{\partial W^{\alpha_r}}{\partial x^i_{\alpha_1 \ldots \alpha_{r-1}}},$$

$$i = \overline{1, n}.$$

**Lemma 3.2.** *If the Lagrangians*

$$L_1 := x^i_{\alpha_1 \ldots \alpha_j} p_i^{\alpha_1 \ldots \alpha_j} - H, \quad L_2 := X^i_{\alpha_1 \ldots \alpha_j} P_i^{\alpha_1 \ldots \alpha_j} - K,$$

*produce the same multi-time Euler-Lagrange PDEs* [*see* $L_2 = L_1 + D_\zeta W^\zeta$,

$$\sum_{r=1}^{k-1} (-1)^r \frac{1}{n(\alpha_1, \ldots, \alpha_r)} D^{r+1}_{\alpha_1 \ldots \alpha_r \zeta} \frac{\partial W^\zeta}{\partial x^i_{\alpha_1 \ldots \alpha_r}}$$

$$= \sum_{r=2}^{k} (-1)^{r+1} \frac{1}{n(\alpha_1, \ldots, \alpha_r)} \frac{1}{n(\alpha_1, \ldots, \alpha_{r-1})} D^r_{\alpha_1 \ldots \alpha_{r-1} \alpha_r} \frac{\partial W^{\alpha_r}}{\partial x^i_{\alpha_1 \ldots \alpha_{r-1}}}, \quad i = \overline{1, n}],$$

*then the change of variables*

$$(x^i, p_i^{\alpha_1}, \ldots, p_i^{\alpha_1 \ldots \alpha_k}, t) \hookrightarrow (X^i, P_i^{\alpha_1}, \ldots, P_i^{\alpha_1 \ldots \alpha_k}, t), \quad i = \overline{1, n},$$

*is a canonical transformation.*

**Proof.** Using Corollary 2.1, the result is obvious.

The vector function $W$ is called the (*higher-order*) *multi-time generating function* of the canonical transformation.

## 4. Conclusion

In the present paper, we have studied (in a mathematical framework governed by higher-order Lagrangians) the relations which appear between two Lagrangians joined by a gauge transformation, the change of variables in Hamiltonian and the generating function. In this way, we have extended and improved the results in Treanţă [8].

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