Research and Communications in Mathematics and Mathematical Sciences
Vol. 3, Issue 2, 2013, Pages 63-74
ISSN 2319-6939
Published Online on January 25, 2014

http://jyotiacademicpress.net

EXISTENCE OF STATIONARY SOLUTIONS FOR SIR MODEL

EL BERRAI IMANE, BOUYAGHROUMNI JAMAL and NAMIR ABDELOUAHED

LAMS (Laboratory of Analysis, Modeling and Simulation)
Department of Mathematics and Computer Sciences
Faculty of Sciences
Hassan II University Mohammedia
Ben M'Sik-Casablanca
Casablanca
Morocco
e-mail: Im.elberrai@gmail.com

Abstract

In the present work, we consider a mathematical modelling of an SIR epidemic model with S the susceptible, "healthy individuals who can catch the disease", I the infected, "those who have the disease and can transmit it", and R the removed "individuals who have had the disease and are now immune to the infection". We focus on a spatiotemporal distribution of healthy and infected populations. We study the existence of stationary solutions to the problem type SIR through Schauder's theorem.

2010 Mathematics Subject Classification: 92B05, 92D25, 37C25.
Keywords and phrases: SIR model, dissemination of epidemics, fixed point.
Communicated by Carlo Bianca.

Received December 6, 2013; Revised December 21, 2013

1. Introduction

Mathematical models of infectious diseases have an important role in epidemiology. They make it possible to predict the evolution of the spread of disease within a population, based on various parameters, such as internal displacement, the evolution of the disease or spatial constraints (see, for example, Capasso and Serio [7]; Hethcote and Tudor [12]; Liu et al. [15, 16]; Hethcote et al. [13]; Hethcote and van den Driessche [14]; Derrick and van den Driessche [9]; Beretta and Takeuchi [3, 4]; Beretta et al. [5]; Ma et al. [17, 18]; Ruan and Wang [20]; Song and Ma [21]; Song et al. [22]; D'Onofrio et al. [19]; Xiao and Ruan [23]). These models can also highlight the impact of vaccination, prevention, and the means used to stop the disease, to test their effectiveness to find an optimal strategy to prevent epidemics. Mathematical epidemiology seems to have grown exponentially starting in the middle of the 20th century (the first edition in 1957 of Baileys book [1] is an important landmark), so that a tremendous variety of models have now been formulated, mathematically analyzed, and applied to infectious diseases. Reviews of the literature [2, 8] show the rapid growth of epidemiology modelling.

We study the SIR model, which divides the population into three epidemiological classes: healthy and potentially infected individuals (S), those who are infected and so infectious (I), and those who have acquired immunity after recovery or death, (R). This type of model, which, for example, models very well the Black Plague in Europe in the fourteenth century, is still used for AIDS nowadays.

The remaining parts of this paper are organized as follows: Section 2 presents the SIR model. In Section 3, we study the behaviour of the epidemic in each location of a territory, by adding a spatial component to the previous model. We study the existence of stationary solutions to the problem type SIR through the theorem of Schauder in Section 4. The last section provides concluding remarks.

2. SIR Model

We study the following SIR model:

$$\begin{cases} \dot{S}(t) = r_c S(t)\left(1 - \dfrac{S(t)}{k}\right) - \dfrac{\alpha S(t) I(t)}{1 + aI(t)}, \\ \dot{I}(t) = \dfrac{\alpha S(t) I(t)}{1 + aI(t)} - \gamma I(t), \\ \dot{R}(t) = \gamma I(t). \end{cases} \tag{1}$$

The model has a susceptible group designated by S, an infected group I, and a recovered group R with permanent immunity, r_c is the intrinsic growth rate of susceptible, k is the carrying capacity of the susceptible in the absence of infective, α is the maximum values of per capita reduction rate of S due to I, a is half saturation constants, and γ is the natural recover rate from infection.

This model is an appropriate one to use under the following assumptions:

(1) The population is fixed.

(2) The only way a person can leave the susceptible group is to become infected. The only way a person can leave the infected group is to recover from the disease. Once a person has recovered, the person received immunity.

(3) Age, sex, social status, and race do not affect the probability of being infected.

3. Spatial Distribution of the SIR Model

Here, we are interested in the spatial distribution of healthy and infected populations, instead of considering them only in their entirety. We work in a model where the healthy and infected individuals move. If we denote by $S(t, x)$ (resp., $I(t, x)$) the density of healthy foxes (infected, respectively) at the abscissa x and time t, these quantities satisfy equations of the type

$$\begin{cases} \dot{S}(t) = \nabla^2(S) + r_c S(t)\left(1 - \dfrac{S(t)}{k}\right) - \dfrac{\alpha S(t)I(t)}{1 + aI(t)}, \\ \dot{I}(t) = \nabla^2(I) + \dfrac{\alpha S(t)I(t)}{1 + aI(t)} - \gamma I(t). \end{cases} \tag{2}$$

Considering that the population moves on a straight line, that is to say we place in dimension 1.

We study the existence of stationary solutions (population status in big time when setting the values of S and I at the edges of a range).

To move to the stationary problem, just look in infinite time, the system does not depend on time then we obtain

$$\begin{cases} \dfrac{\partial^2 S}{\partial x^2} = -r_c S(x)\left(1 - \dfrac{S(x)}{k}\right) + \dfrac{\alpha S(x)I(x)}{1 + aI(x)}, \\ \dfrac{\partial^2 I}{\partial x^2} = \dfrac{-\alpha S(x)I(x)}{1 + aI(x)} + \gamma I(x). \end{cases} \tag{3}$$

To show the existence of stationary solution is going to use the theorem of Schauder for this system for all $x \in [c, d]$,

$$\begin{cases} \dfrac{-\partial^2 S}{\partial x^2} = r_c S(x)\left(1 - \dfrac{S(x)}{k}\right) - \dfrac{\alpha S(x)I(x)}{1 + aI(x)}, & S(c) = \alpha_1,\ S(d) = \beta_1, \\ \dfrac{-\partial^2 I}{\partial x^2} = \dfrac{\alpha S(x)I(x)}{1 + aI(x)} - \gamma I(x), & I(c) = \alpha_2,\ I(d) = \beta_2. \end{cases} \tag{4}$$

4. General Framework of Existence of Stationary Solutions

4.1. Theorem of Schauder

Theorem 4.1. *Let E be a normed vector space, and let $K \subset E$ be a non-empty, compact, and convex set. Then given any continuous mapping $h : K \to K$, there exists $x \in K$ such that $h(x) = x$.*

4.2. Implementation of the integral form and definition of the problem of fixed point

To show the existence of solutions to our equations, we transform the system of differential equations in integral equations to obtain a fixed point problem. For all $x \in [c, d]$, we have

$$\begin{cases} \dfrac{-\partial^2 S}{\partial x^2} = r_c S(x)\left(1 - \dfrac{S(x)}{k}\right) - \dfrac{\alpha S(x) I(x)}{1 + aI(x)}, & S(c) = \alpha_1,\ S(d) = \beta_1, \\ \dfrac{-\partial^2 I}{\partial x^2} = \dfrac{\alpha S(x) I(x)}{1 + aI(x)} - \gamma I(x), & I(c) = \alpha_2,\ I(d) = \beta_2. \end{cases} \tag{5}$$

We put

$$f(S(x),\ I(x)) = r_c S(x)\left(1 - \frac{S(x)}{k}\right) - \frac{\alpha S(x) I(x)}{1 + aI(x)},$$

$$g(S(x),\ I(x)) = \frac{\alpha S(x) I(x)}{1 + aI(x)} - \gamma I(x). \tag{6}$$

We start from the system (5), using (6) then integrating twice, we obtain for all $x \in [c, d]$,

$$\begin{cases} S(x) = \alpha_1 + (x - c)\left(\dfrac{(\beta_1 - \alpha_1) + \int_c^d \int_c^t f(S,\ I) d\delta\, dt}{d - c}\right) - \int_c^x \int_c^t f(S,\ I) d\delta\, dt, \\ I(x) = \alpha_2 + (x - c)\left(\dfrac{(\beta_2 - \alpha_2) + \int_c^d \int_c^t g(S,\ I) d\delta\, dt}{d - c}\right) - \int_c^x \int_c^t g(S,\ I) d\delta\, dt. \end{cases}$$

It is therefore to find $N(S, I) = (S, I)$ solution of problem of fixed point such $N(S, I) = \big(F(S, I), G(S, I)\big)$, which is equivalent to find (S, I) solution of problem of fixed point such $F(S, I) = S$ and $G(S, I) = I$, defined by:

For all $x \in [c, d]$,

$$\begin{cases} F(S, I)(x) = \alpha_1 + (x - c)\left(\dfrac{(\beta_1 - \alpha_1) + \int_c^d \int_c^t f(S, I)d\delta\, dt}{d - c}\right) - \int_c^x \int_c^t f(S, I)d\delta\, dt, \\ G(S, I)(x) = \alpha_2 + (x - c)\left(\dfrac{(\beta_2 - \alpha_2) + \int_c^d \int_c^t g(S, I)d\delta\, dt}{d - c}\right) - \int_c^x \int_c^t g(S, I)d\delta\, dt. \end{cases}$$

4.3. Hypothesis testing

• **Choice of assumptions on** $f(S(x), I(x))$ **and** $g(S(x), I(x))$:

Let $S \in [s_1, s_2]$ and $I \in [i_1, i_2]$, we defined on $U = [s_1, s_2][i_1, i_2]$ the functions *f* and *g* as follows:

For all $x \in [c, d]$,

$$f(S(x), I(x)) = r_c S(x)\left(1 - \frac{S(x)}{k}\right) - \frac{\alpha S(x) I(x)}{1 + aI(x)},$$

$$g(S(x), I(x)) = \frac{\alpha S(x) I(x)}{1 + aI(x)} - \gamma I(x).$$

Proposition 4.2. *f and g are bounded functions in U, then there exists* M_1 *and* M_2 *such that* $|f| \leq M_1$, $|g| \leq M_2$. *With*

$$M_1 = r_c s_2\left(1 + \frac{s_2}{k}\right) + \frac{\alpha s_2 i_2}{1 + a i_1};$$

$$M_2 = \frac{\alpha s_2 i_2}{1 + a i_1} + \gamma i_2.$$

We will show that the functions *f* and *g* satisfy the Lipschitz condition on interval *U*. Let $(S, S') \in [s_1, s_2]^2$ and $(I, I') \in [i_1, i_2]^2$.

$$|f(S, I) - f(S', I')| = \left| r_c S\left(1 - \frac{S}{k}\right) - \frac{\alpha SI}{1 + aI} - r_c S'\left(1 - \frac{S'(x)}{k}\right) + \frac{\alpha S' I'}{1 + aI'} \right|$$

$$\leq r_c|S - S'| + \frac{r_c}{k}|S'^2 - S^2| + \left|\alpha\left(\frac{S'I'}{1 + aI'} - \frac{SI}{1 + aI}\right)\right|$$

$$\leq r_c|S - S'| + \frac{r_c}{k} 2s_2|S - S'| + \alpha\left|\frac{I(S - S')}{1 + aI} + \frac{S'I}{1 + aI} - \frac{S'I'}{1 + aI'}\right|$$

$$\leq r_c|S - S'| + \frac{r_c}{k} 2s_2|S - S'| + \alpha\left|\frac{I}{1 + aI}\right||S - S'| + \alpha\left|\frac{S'(I - I')}{(1 + aI)(1 + aI')}\right|$$

$$\leq r_c|S - S'| + \frac{r_c}{k} 2s_2|S - S'| + \alpha|S - S'| + \alpha\frac{s_2}{(1 + ai_1)^2}|I - I'|$$

$$\leq \left(r_c + \frac{2r_c s_2}{k} + \alpha\right)|S - S'| + \alpha\frac{s_2}{(1 + ai_1)^2}|I - I'|$$

$$\leq \max\left((r_c + \frac{2r_c s_2}{k} + \alpha),\ \alpha\frac{s_2}{(1 + ai_1)^2}\right)\left(|S - S'| + |I - I'|\right)$$

$$\leq k_1\left(|S - S'| + |I - I'|\right).$$

With

$$k_1 = \max\left((r_c + \frac{2r_c s_2}{k} + \alpha),\ \alpha\frac{s_2}{(1 + ai_1)^2}\right).$$

So the function f is k_1 Lipschitz on U.

By same reasoning, the function g is k_2 Lipschitz on U with $k_2 = \max\left(\alpha,\ \alpha\frac{\alpha s_2}{(1 + ai_1)^2} + \gamma\right)$.

• **Determination of convex K:**

We define K as follows:

$$K = \{S \geq 0,\ \| S \|_{L^2]c, d[} \leq C\}.$$

• **Determination of C:**

Let $B \in C^0([c, d])$, ϕ an affine function with $\phi(c) = \alpha$, $\phi(d) = \beta$ and A solution of system

$$\begin{cases} -A'' = B, \\ A(c) = \alpha, \; A(d) = \beta. \end{cases}$$

$(A - \phi)$ satisfies the following system:

$$\begin{cases} -(A - \phi)'' = B, \\ (A - \phi)(c) = 0, \; (A - \phi)(d) = 0. \end{cases}$$

Thus, by multiplying $(A - \phi)$ and integrating from c to d, we obtain

$$\int_c^d B(A - \phi) = \int_c^d -(A - \phi)''(A - \phi) = \|(A - \phi)'^2\|_{L^2]c, d[}.$$

Using Poincare's inequality and Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned} \|(A - \phi)\|_{L^2]c, d[} &\leq (d - c)^2 \|(A - \phi)'\|_{L^2]c, d[} \\ &= (d - c)^2 \int_c^d B(A - \phi) \\ &\leq (d - c)^2 \| B \|_{L^2]c, d[} + \| \phi \|_{L^2]c, d[} \\ &\leq (d - c)^2 \| B \|_{L^2]c, d[} + \max(|\alpha|, |\beta|)\sqrt{d - c}. \end{aligned}$$

So

$$\begin{aligned} \| F(S, I) \|_{L^2]c, d[} &\leq (d - c)^2 \| f(S, I) \|_{L^2]c, d[} + \max(|\alpha|, |\beta|)\sqrt{d - c} \\ &\leq (d - c)^{\frac{5}{2}} M_1 + \max(|\alpha|, |\beta|)\sqrt{d - c}, \\ \| G(S, I) \|_{L^2]c, d[} &\leq (d - c)^2 \| g(S, I) \|_{L^2]c, d[} + \max(|\alpha|, |\beta|)\sqrt{d - c} \\ &\leq (d - c)^{\frac{5}{2}} M_2 + \max(|\alpha|, |\beta|)\sqrt{d - c}. \end{aligned}$$

So

$$C = (d - c)^{\frac{5}{2}} \max(M_1, M_2) + \max(|\alpha|, |\beta|)\sqrt{d - c}. \quad (7)$$

- (F, G) **continuous:**

Let $(S, I) \in K^2$, $(S', I') \in K^2$ and $x \in [c, d]$.

$|F(S, I) - F(S', I')|$

$$= \left| \int_c^x \int_c^t f(S, I) - f(S', I') dn\, dt - \frac{x - c}{d - c} \left(\int_c^d \int_c^t f(S, I) - f(S', I') d\delta\, dt \right) \right|$$

$$\leq \int_c^d \int_c^d |f(S, I) - f(S', I')| d\delta\, dt + \frac{d - c}{d - c} \left(\int_c^d \int_c^t |f(S, I) - f(S', I')| d\delta\, dt \right)$$

$$\leq k_1(d - c) \int_c^d |S - S'| - |I - I'| d\delta + \int_c^d \int_c^d |f(S, I) - f(S', I')| d\delta$$

$$\leq 2k_1(d - c) \sqrt{\int_c^d 1\, d\delta} \left(\sqrt{\int_c^d |S - S'|^2 d\delta} + \sqrt{\int_c^d |I - I'|^2 d\delta} \right)$$

$$\leq 2k_1(d - c)^{\frac{3}{2}} (\| S - S' \|_{L^2]c, d[} + \| I - I' \|_{L^2]c, d[}).$$

Therefore (F, G) is $2k_1(d - c)^{\frac{3}{2}}$ Lipschitz, so (F, G) is continuous.

- (F, G) **compact:**

Let $(S, I) \in K^2$ and $(x, y) \in [c, d]^2$.

$|F(S, I)(x) - F(S, I)(y)|$

$$= \left| \int_x^y \int_c^t f(S, I) d\delta\, dt + (x - y) \left(\frac{(\beta_1 - \alpha_1) + \int_c^d \int_c^t f(S, I) d\delta\, dt}{d - c} \right) \right|$$

$$\leq \left|\int_x^y \int_c^t f(S,\ I) d\delta\, dt\right| + \left|(x-y)\left(\frac{(\beta_1 - \alpha_1) + \int_c^d \int_c^t f(S,\ I) d\delta\, dt}{d-c}\right)\right|$$

$$\leq \left|\int_x^y \int_c^t f(S,\ I) d\delta\, dt\right| + \frac{|x-y|}{d-c}\left|\left((\beta_1 - \alpha_1) + \int_c^d \int_c^t f(S,\ I) d\delta\, dt\right)\right|$$

$$\leq |x-y||d-c| \| f \|_\infty + \frac{|x-y|}{d-c}(|\beta_1 - \alpha_1| + |(d-c)^2| \| f \|_\infty)$$

$$\leq |x-y|\left(2|d-c| \| f \|_\infty + \frac{|\beta_1 - \alpha_1|}{d-c}\right).$$

Therefore $F(S,\ I)$ is $\left(2|d-c| \| f \|_\infty + \frac{|\beta_1 - \alpha_1|}{d-c}\right)$ Lipschitz so $F(K)^2$ is equi-continuous and we know that $F(K)^2$ is bounded. Inspiring of Ascoli's theorem, the $F(K)^2$ is a compact on $L^2]c,\ d[$.

5. Conclusion

The SIR is a nonlinear dynamical system which displays complicated behaviours, which are difficult to understand in intuitive terms. Moreover, these complicated behaviours are relevant in order to explain the epidemiology of infectious diseases in humans, as studies comparing the behaviour of the SIR and its variants of it to surveillance data have shown [6, 10]. We prove the existence of the stationary solutions of the considered problem by using the Schauder's theorem.

The existence of stationary solutions of the considered problem, can explain, on one hand that the model is well-posed, and on the other hand opens the way for the numerical analysis and numerical simulation. El Berrai et al. [11] was designed for resolution of the differential system (5) by using the finite difference approach based on explicit Euler schema. Thus, the study of the stability of stationary solutions will be the subject of future work.

References

[1] N. T. J. Bailey, The Mathematical Theory of Infectious Diseases, 2nd Edition, Hafner, New York, 1975.

[2] N. Becker, The use of epidemic models, Biometrics 35 (1978), 295-305.

[3] E. Beretta and Y. Takeuchi, Global stability of an SIR epidemic model with time delay, J. Math. Biol. 33 (1995), 250-260.

[4] E. Beretta and Y. Takeuchi, Convergence results in SIR epidemic model with varying population sizes, Nonlinear Anal. 28 (1997), 1909-1921.

[5] E. Beretta, T. Hara, W. Ma and Y. Takeuchi, Global asymptotic stability of an SIR epidemic model with distributed time delay, Nonlinear Anal. 47 (2001), 4107-4115.

[6] B. M. Bolker and B. T. Grenfell, Chaos and biological complexity in measles dynamics, Proc. Roy. Soc. Lon. B 251 (1993), 75-81.

[7] V. Capasso and G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model, Math. Biosci. 42 (1978), 43-61.

[8] C. Castillo-Chavez, ed., Mathematical and Statistical Approaches to AIDS Epidemiology, Lecture Notes in Biomath. 83, Springer-Verlag, Berlin, 1989.

[9] W. R. Derrick and P. van den Driessche, A disease transmission model in nonconstant population, J. Math. Biol. 31 (1993), 495-512.

[10] P. Rohani Earn, B. M. Bolker and B. T. Grenfell, A simple model for complex dynamical transitions in epidemics, Science 287 (2000), 667-670.

[11] I. EL Berrai, J. Bouyaghroumni and A. Namir, Dissemination of epidemic for SIR model, Applied Mathematical Sciences 7(136) (2013), 6793-6800.

[12] H. W. Hethcote and D. W. Tudor, Integral equation models for endemic infectious diseases, J. Math. Biol. 9 (1980), 37-47.

[13] H. W. Hethcote, M. A. Lewis and P. van den Driessche, An epidemiological model with delay and a nonlinear incidence rate, J. Math. Biol. 27 (1989), 49-64.

[14] H. W. Hethcote and P. van den Driessche, Some epidemiological model with nonlinear incidence, J. Math. Biol. 29 (1991), 271-287.

[15] W. M. Liu, H. W. Hethcote and S. A. Levin, Dynamical behaviour of epidemiological models with nonlinear incidence rates, J. Math. Biol. 25 (1987), 359-380.

[16] W. M. Liu, S. A. Levin and Y. Iwasa, Influence of nonlinear incidence rates upon the behaviour of SIRS epidemiological models, J. Math. Biol. 23 (1986), 187-204.

[17] W. Ma, M. Song and Y. Takeuchi, Global stability of an SIR epidemic model with time delay, Appl. Math. Lett. 17 (2004), 1141-1145.

[18] W. Ma, Y. Takeuchi, T. Hara and E. Beretta, Permanence of an SIR epidemic model with distributed time delays, Tohoku Math. J. 54 (2002), 581-591.

[19] A. D'Onofrio, P. Manfredi and E. Salinelli, Bifurcation thresholds in an SIR model with information-dependent vaccination, Mathematical Modelling of Natural Phenomena 2 (2007), 23-38.

[20] S. Ruan and W. Wang, Dynamical behaviour of an epidemic model with nonlinear incidence rate, J. Differ. Equations 188 (2003), 135-163.

[21] M. Song and W. Ma, Asymptotic properties of a revised SIR epidemic model with density dependent birth rate and tie delay, Dynamic of Continuous, Discrete and Impulsive Systems 13 (2006), 199-208.

[22] M. Song, W. Ma and Y. Takeuchi, Permanence of a delayed SIR epidemic model with density dependent birth rate, J. Comput. Appl. Math. 201 (2007), 389-394.

[23] D. Xiao and S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, Math. Biosci. 208 (2007), 419-429.

■