CHARACTERIZATION IN TERMS OF MEASURE OF LACUNARY UNIFORM STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract

In the [3] is proven that almost every, in terms of measure $P_A$, subsequence $S(x)$ of double sequence $S$ converges to $L$ in the Pringsheim’s sense, if and only if sequence $S$ uniformly statistically converges to $L$. In this paper, it is proven that analogue is valid and for lacunary uniformly statistical convergence. Almost every, in terms of measure $P_A$, subsequence $S(x)$ of double sequence $S$ converges to $L$ in the Pringsheim’s sense, if and only if sequence $S$ lacunary uniformly statistically converges to $L$.

This is not true for measure $P$.

Almost every, in terms of measure $P$, subsequence $S(x)$ of double sequence $S$ of 0’s and 1’s is not almost uniformly statistically convergent, if is sequence $S$ lacunary uniformly statistically convergent and divergent in the Pringsheim’s sense.

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1. Introduction

The concept of the statistical convergence of a sequences of real numbers was introduced by Fast [10]. Furthermore, Gökhan et al. [13] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued function. Dündar and Altay [5-9] investigated the relation between $I$-convergence of double sequences. Fridy and Orhan [12] have studied lacunary statistical convergence of single sequences. Patterson and Savaş in [14] defined the lacunary statistical analogue for double sequences. Now, we recall that the definitions of concepts of ideal convergence and basic concepts [1, 2, 11].

The sequence $S_{ij}$ of real numbers converges to $L$ in the Pringsheim’s sense, if for $\forall \varepsilon > 0$, $\exists K > 0$ such that

$$|S_{ij} - L| \leq \varepsilon, \forall i, j \geq K.$$ 

We write $\lim_{i, j \to \infty} S_{ij} = L$.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K_{nm}$ be the number of $(i, j) \in K$ such that $i \leq n$, $j \leq m$. If

$$d_2(K) = \lim_{n,m \to \infty} \frac{K_{nm}}{nm},$$

in the Pringsheim’s sense then, we say that $K$ has double natural density. Let is sequence $S_{ij}$ of real numbers and $\varepsilon > 0$. Let

$$A(\varepsilon) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\}.$$ 

The sequence $S = S_{ij}$ statistically converges to $L \in \mathbb{R}$ if $d_2(A(\varepsilon)) = 0$ for $\forall \varepsilon > 0$. 
We write \( st - \lim S_{ij} = L \). Let is set \( X \neq \emptyset \). A class \( I \) of subsets of \( X \) is said to be an *ideal in \( X \) provided the following statements hold:

(i) \( \emptyset \in I \),

(ii) \( A, B \in I \Rightarrow A \cup B \in I \),

(iii) \( A \in I, B \subset A \Rightarrow B \in I \).

The ideal is called *nontrivial* if \( I \neq \{\emptyset\} \) and \( X \in I^c \). A nontrivial ideal \( I \) is called *admissible* if it contains all the singleton sets. A nontrivial ideal \( I \) on \( \mathbb{N} \times \mathbb{N} \) is called *strongly admissible* if \( \{i\} \times \mathbb{N} \) and \( \mathbb{N} \times \{i\} \) belong to \( I \) for each \( i \in \mathbb{N} \).

A nonempty family \( F \) of subsets of a set \( X \) is called a *filter* if

(i) \( \emptyset \in F^c \),

(ii) \( A, B \in F \Rightarrow A \cap B \in F \),

(iii) \( A \in F, A \subset B \Rightarrow B \in F \).

In this paper, the focus is put on ideal \( I_u \subset 2^{\mathbb{N} \times \mathbb{N}} \) defined by: subset \( A \) belongs to the \( I_u \) if

\[
\lim_{p,q \to \infty} \frac{1}{pq} |\{i < p, j < q : (n + i, m + j) \in A\}| = 0,
\]

uniformly on \( n, m \in \mathbb{N} \) in the Pringsheim’s sense. That is subset \( A \) of the set \( \mathbb{N} \times \mathbb{N} \) is uniformly density zero.

The sequence \( S = S_{ij} \) uniformly statistically converges to \( L \) if for any \( \varepsilon > 0 \)

\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\} \in I_u.
\]
That is sequence $S = S_{ij}$ uniformly statistically converges to $L$ if
forall $\varepsilon, \varepsilon' > 0$, $\exists K > 0$ such that

$$\frac{1}{pq} |\{i < p, j < q : |S_{n+i,m+j} - L| \geq \varepsilon\}| < \varepsilon', \forall p, q \geq K, \forall n, m \in \mathbb{N}.$$ 

We write $\text{Ust - lim} S_{ij} = L$.

We denote with $X$ a set of all double sequences of 0's and 1's, i.e.,

$$X = \{x = x_{ij} : x_{ij} \in \{0, 1\}, i, j \in \mathbb{N}\}.$$ 

Let sequence $S = S_{ij}$ and $x \in X$. Then with $S(x)$ we denote a sequence defined following way:

$$S_{ij}(x) = S_{ij}, \text{ for } x_{ij} = 1.$$ 

The mapping $x \rightarrow S(x)$ is a bijection of the set $X$ to a set of all subsequences of the sequence $S$.

Then, under the Lebesgue measure on the set of all subsequences of the sequence $S$ consider Lebesgue measure on the set $X$.

Let $\beta$ smallest $\sigma$-algebra subsets of the set $X$ which contains of subsets in the form of:

$$\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1, x_{n_2m_2} = a_2, \ldots, x_{n_km_k} = a_k\},$$

$$a_1, \ldots, a_k \in \{0, 1\}, k \in \mathbb{N}.$$ 

There is a unique Lebesgue measure $P$ on the set $X$ for which the following applies:

$$P(\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1, x_{n_2m_2} = a_2, \ldots, x_{n_km_k} = a_k\}) = \frac{1}{2^k}.$$ 

The subsequence $S(x)$ of sequence $S$ uniformly statistically converges to $L$ if $\forall \varepsilon, \varepsilon' > 0$, $\exists K > 0$ such that for $\forall p, q \geq K$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have
\[
\left\{\frac{\left|i < p, j < q : |S_{n+i,m+j} - L| \geq \varepsilon, x_{n+i,m+j} = 1\right|}{\left|i < p, j < q : x_{n+i,m+j} = 1\right|}\right\} \leq \varepsilon'.
\]

We write \( \text{Ust} - \lim S_{ij}(x) = L \).

By a lacunary sequence, we mean an increasing sequence \( \Theta = (k_r) \) such that
\[
k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.
\]

Let
\[
I_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_1\},
\]
\[
I_2 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_2\} \setminus I_1, \ldots,
\]
\[
I_r = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_r\} \setminus (I_{r-1} \cup I_{r-2} \cup \ldots \cup I_1), \text{ for } \forall r \in \mathbb{N}.
\]

The sequence \( S_{ij} \) lacunary statistically converges to \( L \) if \( \forall \varepsilon > 0 \), we have
\[
\lim_{r \to \infty} \frac{1}{|I_r|} \left|\{(i, j) \in I_r : |S_{ij} - L| \geq \varepsilon\}\right| = 0.
\]

We write \( S_\Theta - \lim S_{ij} = L \).

Fridy proved that if \( S = S_i \) sequence of real numbers and \( \Theta = (k_r) \) lacunary sequence such that
\[
1 < \lim inf \frac{k_r}{k_{r-1}} \leq \lim sup \frac{k_r}{k_{r-1}} < \infty.
\]

Then, sequence \( S = S_i \) statistically convergent if and only if it lacunary statistically convergent.

Let \( S = S_{ij} \) double sequence of real numbers and \( \Theta = (k_r) \) lacunary sequence of natural numbers.
A sequence $S = S_{ij}$ lacunary uniformly statistically converges to real number $L$ if $\forall \varepsilon, \varepsilon' > 0$, $\exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N}$, we have

$$\frac{1}{|I_r|} \left| \{(i, j) \in I_r : |S_{n+i, m+j} - L| \geq \varepsilon\} \right| \leq \varepsilon'. $$

We write $\lim_{\text{UST}} S_{ij} = L$.

The subset $A$ of the set $\mathbb{N} \times \mathbb{N}$ is lacunary uniformly density zero if $\forall \varepsilon > 0$, $\exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N}$, we have

$$\frac{1}{|I_r|} \left| \{(i, j) \in I_r : (n + i, m + j) \in A\} \right| \leq \varepsilon. $$

2. New Results

Not almost every, in terms of $P$, subsequence $S(x)$ of double sequence $S$ is convergent to $L$ in the Pringsheim’s sense if $S$ converges to $L$ lacunary uniformly statistically.

**Example.** Let be $\Theta = (k_r)$ lacunary sequence and $A \subset \mathbb{N} \times \mathbb{N}$ lacunary uniformly density zero such that for $\forall N \in \mathbb{N}$, $\exists (i, j) \in A$ and $i, j \geq N$.

Let the sequence $S = (S_{ij})$ defined as

$$S_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

Then, $\forall \varepsilon > 0$, $\forall n, m \in \mathbb{N}$, the following is valid:

$$\frac{1}{|I_l|} \left| \{(i, j) \in I_l : |S_{n+i, m+j} - 1| \geq \varepsilon\} \right|$$

$$= \frac{1}{|I_l|} \left| \{(i, j) \in I_l : (n + i, m + j) \in A\} \right| \to 0 \text{ for } l \to \infty.$$
Respectively, \(\text{Ust}_\Theta - \lim S_{ij} = 1\). Let

\[
B = \bigcap_{M=1}^{\infty} \bigcup_{i,j \geq M} \{x \in X : x_{ij} = 1, (i, j) \in A\}.
\]

\[
\sum_{i,j \geq M, (i,j) \in A} P(\{x \in X : x_{ij} = 1\}) = \sum_{i,j \geq M, (i,j) \in A} \frac{1}{2} = \infty.
\]

Due to second part of Borel-Cantelli lemma, \(P(B) = 1\).

Since subsequence \(S(x)\) of \(S\) does not converge to 1 in the Pringsheim’s sense if and only if \(x \in B\), it,

\[
P(\{x \in X : \lim_{i,j \to \infty} S_{ij}(x) = 1 \text{ in the Pringsheim’s sense}\}) = 0.
\]

Let \(A \subset \mathbb{N} \times \mathbb{N}\). There is a unique measure \(P_A\) on \(X\) with the property:

\[
P_A(\{x \in X : x_{ij} = 1\}) = \begin{cases} 
\frac{1}{2}, & (i, j) \notin A, \\
\frac{1}{2^{i+j}}, & (i, j) \in A,
\end{cases}
\]

\[
P_A(\{x \in X : x_{i_1,\cdot} = a_1, \ldots, x_{i_k,\cdot} = a_k\}) = P_A(\{x \in X : x_{i_1,\cdot} = a_1\}) \cdots P_A(\{x \in X : x_{i_k,\cdot} = a_k\}).
\]

Analogue theorem is valid: Let the sequence \(S = (S_{ij})\) be divergent in the Pringsheim’s sense. Then, \(S\) uniformly statistically converges to \(L\) if and only if \(\exists A \subset \mathbb{N} \times \mathbb{N}\) uniformly density zero such that

\[
P_A(\{x \in X : \lim_{i,j \to \infty} S_{ij}(x) = L \text{ in the Pringsheim’s sense}\}) = 1.
\]

**Theorem 2.1.** Let the sequence \(S = (S_{ij})\) divergent in the Pringsheim’s sense. Then, the sequence \(S\) lacunary uniformly statistically converges to \(L\) if and only if \(\exists A \subset \mathbb{N} \times \mathbb{N}\) lacunary uniformly density zero such that

\[
P_A(\{x \in X : \lim_{i,j \to \infty} S_{ij}(x) = L \text{ in the Pringsheim’s sense}\}) = 1.
\]
Proof. Because of lemma the following is valid: Let is \( \lim_{\omega} \text{Ust}_\omega \) \( S_{ij} = L \), then \( \exists A \subset \mathbb{N} \times \mathbb{N} \) lacunary uniformly density zero such that the subsequence \( S(y) \) of \( S \) converges to \( L \) in the Pringsheim’s sense for
\[
y_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}
\]
Not generalizing we can assume that \( L \) is not a point accumulation of the subsequence \( S(x) \) for
\[
x_{ij} = \begin{cases} 1, & (i, j) \in A, \\ 0, & (i, j) \notin A. \end{cases}
\]
Hence, the subsequence \( S(z) \) converges to \( L \) in the Pringsheim’s sense if and only if \( \exists M \in \mathbb{N} \) such that
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : z_{ij} = 1, i, j \geq M\} \cap A = \emptyset.
\]
Let
\[
B_M = \{x \in X : x_{ij} = 1, i, j \geq M, (i, j) \in A\}, \quad B = \bigcap_{M=1}^{\infty} B_M.
\]
Then, \( \forall M \in \mathbb{N} \), is,
\[
P_A(B) \leq P_A(B_M) = \sum_{i,j \geq M, (i,j) \notin A} \frac{1}{2^{i+j}} \leq \sum_{i,j \geq M} \frac{1}{2^{i+j}} = \frac{1}{2^{2M-2}}.
\]
Hence, \( P_A(B) = 0 \). Since the set \( B \) is a set of all \( x \in X \) for which \( S(x) \) does not converge to \( L \) in the Pringsheim’s sense. It,
\[
P_A(\{x \in X : \lim_{i,j \to \infty} S_{ij}(x) = L \text{ in the Pringsheim’s sense}\}) = 1.
\]
Let the sequence \( S \) not be lacunary uniformly statistically convergent and let \( A \subset \mathbb{N} \times \mathbb{N} \) lacunary uniformly density zero. Then, due to the lemma, the subsequence \( S(x) \) is divergent in the Pringsheim's sense for

\[
x_{ij} = \begin{cases} 
1, & (i, j) \notin A, \\
0, & (i, j) \in A.
\end{cases}
\]

The following cases can be presented:

(a) \( \exists (n_k), (m_k), n_k \not\rightarrow, m_k \not\rightarrow, (n_k, m_k) \notin A, S_{n_km_k} \geq k \) for \( \forall k \),

(b) \( \exists (n_k), (m_k), n_k \not\rightarrow, m_k \not\rightarrow, (n_k, m_k) \notin A, S_{n_km_k} \leq -k \) for \( \forall k \),

(c) \( \exists (n_k^n), (m_k^n), \exists (n_k^2), (m_k^2), (n_k^1, m_k^1), (n_k^2, m_k^2) \notin A, S_{n_k^1m_k^1} \leq \lambda < \mu \leq S_{n_k^2m_k^2} \). It follows:

(a) \( \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_km_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty \),

(b) \( \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_km_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty \),

(c) \( \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^1m_k^1} = 1\}) = \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^1m_k^2} = 1\}) = \infty \).

Then due to second part of Borel-Cantelli lemma the following is valid:

(a) \( P_A(\{x \in X : x_{n_km_k} = 1 \text{ for infinite } k\}) = 1 \),

(b) \( P_A(\{x \in X : x_{n_km_k} = 1 \text{ for infinite } k\}) = 1 \),

(c) \( P_A(\{x \in X : x_{n_k^1m_k^1} = x_{n_k^1m_k^2} = 1 \text{ for infinite } k\}) = 1 \).

It follows

\[
P_A(\{x \in X : S(x) \text{ divergent in the Pringsheim’s sense}\}) = 1.
\]

Hence,

\[
P_A(\{x \in X : S(x) \text{ convergent in the Pringsheim’s sense}\}) = 0.
\]
Lemma 2.2 ([4]). Let is $\Theta = (k_r)$ lacunary sequence and $S = S_{ij}$ double sequence. Then, $\text{Ust}_\Theta - \lim S_{ij} = L$ if and only if $\exists A \subset \mathbb{N} \times \mathbb{N}$ lacunary uniformly density zero and $\lim_{i,j \to \infty} S_{ij}(x) = L$, in the Pringsheim’s sense, for

$$x_{ij} = \begin{cases} 1, & (i, j) \not\in A, \\ 0, & (i, j) \in A. \end{cases}$$

**Proof.** Let is $\text{Ust}_\Theta - \lim S_{ij} = L$. Then there is a sequence of natural numbers $(u_r)_{r=2}^\infty$ such that for $\forall l \geq u_r$ and $\forall n, m \in \mathbb{N}$, we have

$$\frac{1}{|I_l|} \left| \left\{(i, j) \in I_l : |S_{n+i,m+j} - L| \geq \frac{1}{r} \right\} \right| \leq \frac{1}{r}.$$

Let

$$A = \bigcup_{r=2}^\infty \bigcup_{n,m=1}^\infty \left\{(n+i, m+j) : (i, j) \in \bigcup_{l=u_r}^{u_{r+1}-1} I_l, |S_{n+i,m+j} - L| \geq \frac{1}{r} \right\}.$$

We define $x \in X$ the following way:

$$x_{ij} = \begin{cases} 1, & (i, j) \not\in A, \\ 0, & (i, j) \in A. \end{cases}$$

For $\forall \varepsilon > 0$, $\exists r_0 \in \mathbb{N}$ such that for $\forall r \geq r_0$ we have $\frac{1}{r} \leq \varepsilon$. From the definition of the sequence $x$, such that for $l \geq u_{r_0}$ and $\forall n, m \in \mathbb{N}$ provided that $x_{n+i,m+j} = 1$, we have

$$|S_{n+i,m+j}(x) - L| = |S_{n+i,m+j} - L| \leq \varepsilon.$$

Hence, for $\forall i, j \geq k_{u_{r_0}}$ and $\forall i, j \in \mathbb{N}$ provided that $x_{ij} = 1$, we have

$$|S_{ij}(x) - L| \leq \varepsilon.$$
Hence, the subsequence $S(x)$ converges to $L$, in the Pringsheim’s sense.

For $\forall l \leq u_0$ and $\forall n, m \in \mathbb{N}$ valid

$$\frac{1}{|I_l|}||\{(i, j) \in I_l : (n + i, m + j) \in A]\|$$

$$= \frac{1}{|I_l|}||\{(i, j) \in I_l : |S_{n+i,m+j} - L| \geq \frac{1}{r}\}| \leq \frac{1}{r} \leq \varepsilon.$$  

Hence,

$$\lim_{l \to \infty} \frac{1}{|I_l|}||\{(i, j) \in I_l : (n + i, m + j) \in A]\| = 0,$$  

uniformly for $\forall n, m \in \mathbb{N}$.

We assume that there is a subset $A$ of the set $\mathbb{N} \times \mathbb{N}$ such that

$$\lim_{l \to \infty} \frac{1}{|I_l|}||\{(i, j) \in I_l : (n + i, m + j) \in A]\| = 0,$$  

uniformly for $\forall n, m \in \mathbb{N}$

and $\lim_{i,j \to \infty} S_{ij}(x) = L$, in the Pringsheim’s sense, for

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

For $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for $\forall n, m \geq N$, we have

$$|S_{nm}(x) - L| \leq \varepsilon.$$  

For $\forall l \in \mathbb{N}$ such that $k_{l-1} > N$. Then,

$$\frac{1}{|I_l|}||\{(i, j) \in I_l : |S_{n+i,m+j} - L| \geq \varepsilon\}|$$

$$= \frac{1}{|I_l|}||\{(i, j) \in I_l : n + i < N \lor m + j < N, |S_{n+i,m+j} - L| \geq \varepsilon\}|$$

$$+ \frac{1}{|I_l|}||\{(i, j) \in I_l : n + i, m + j \geq N, |S_{n+i,m+j} - L| \geq \varepsilon\}|.$$
\[
\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|} \left| \{(i, j) \in I_l : n + i, m + j \geq N, (n + i, m + j) \in A \} \right|
\]

\[
\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|} \left| \{(i, j) \in I_l : (n + i, m + j) \in A \} \right|
\]

Obviously, the first summation converges to zero uniformly on \( n, m \in \mathbb{N} \).
The second summation converges to zero uniformly on \( n, m \in \mathbb{N} \) due to the assumption.

So, \( \text{ust}_\Theta - \lim S_{ij} = L \).

**Theorem 2.3.** Let \( \Theta = (k_r) \) lacunary sequence. Then, almost every double sequence of 0's and 1's is not lacunary uniformly statistically convergent.

**Proof.** Let

\[
A_n^r = \{x \in X : x_{n+i, n+j} = 1, (i, j) \in I_r \}.
\]

Since \( P(A_n^r) = \frac{1}{2|I_r|} \), \( \forall n \in \mathbb{N} \), it is

\[
\sum_{n=1}^{\infty} P(A_n^r) = \sum_{n=1}^{\infty} \frac{1}{2|I_r|} = +\infty.
\]

Since \( A_n^k \) are independent, based on the second part of Borel-Cantelli lemma:

\[
P\left( \limsup_{n} A_n^r \right) = 1.
\]

We denote \( A^r = \limsup_n A_n^r \), \( A = \limsup_r A^r \). Since \( P(A^r) = 1 \), \( \forall r \in \mathbb{N} \),

it is \( \sum_{r=1}^{\infty} P(A^r) = +\infty. \)
Due to second part of Borel-Cantelli lemma, it follows $P(A) = 1$. Let
is $x \in A$ then, for $\forall r_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $\exists r > r_0$ such that $x \in A_n^r$. It follows
$\forall x \in A$ does not converge lacunary uniformly statistically to 0.

Completely analogously, almost every $x \in X$ does not converge
lacunary uniformly statistically to 1.

Every lacunary uniformly statistically convergent sequence $x \in X$
converges 0 or 1. It follows
$P(\{x \in X : x = (x_{ij}) \text{ convergent lacunary uniformly statistically} \}) = 0$.

**Definition 2.4.** The subsequence $S(x)$ of sequence $S$ lacunary
uniformly statistically converges to $L$ if $\forall \epsilon, \epsilon' > 0$, $\exists K > 0$ such that for
$\forall r \geq K$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|(i, j) \in I_r : |S_{n+i,m+j} - L| \geq \epsilon, x_{n+i,m+j} = 1|}{|(i, j) \in I_r : x_{n+i,m+j} = 1|} \leq \epsilon'.$$

We write $\text{Ust}_\Theta - \lim S_{ij}(x) = L$.

Almost every, in terms of measure $P$, subsequence of uniformly
statistically convergent double sequence $S$ is uniformly statistically
convergent. This analogue is valid also for lacunary uniform statistical
convergence double sequences of 0’s and 1’s.

**Theorem 2.5.** Let is $S_{ij}$ sequence of 0’s and 1’s that is convergent
lacunary uniformly statistically and divergent in the Pringsheim’s sense. Let
is $\Theta = (k_r)$ lacunary sequence. Then,

$P(\{x \in X : S_{ij}(x) \text{ convergent lacunary uniformly statistically} \}) = 0$.

**Proof.** Since a sequence $S_{ij}$ convergent lacunary uniformly statistically,
it is,

$$\text{Ust}_\Theta - \lim S_{ij} = 1 \text{ or } 0.$$
Suppose it is \( \lim_{i,j \to \infty} S_{ij} = 1 \) and \( \lim_{i,j \to \infty} S_{ij} \neq 1 \) in the Pringsheim's sense. Then, there exists infinite subsequence \((k_r)\) of the sequence \((k_i)\) such that for \( \forall l \in \mathbb{N}, \exists (i, j) \in I_{k_l} \) we have \( S_{ij} = 0 \). Not generalizing we can assume that for \( \forall r \in \mathbb{N}, \exists (i, j) \in I_r \), we have \( S_{ij} = 0 \). Let

\[
A^r_n = \{ x \in X : x_{n+i,n+j} = S'_{n+i,n+j}, (i,j) \in I_r \},
\]

where is

\[
S'_{ij} = \begin{cases} 
1, & S_{ij} = 0, \\
0, & S_{ij} = 1.
\end{cases}
\]

Since \( P(A^r_n) = \frac{1}{2|I_r|} \), it is

\[
\sum_{n=1}^{\infty} P(A^r_n) = \sum_{n=1}^{\infty} \frac{1}{2|I_r|} = +\infty.
\]

Since \( A^r_n \) are independent, based on the second part of Borel-Cantelli lemma:

\[
P\left( \limsup_{n} A^r_n \right) = 1.
\]

We denote \( A^r = \limsup_{n} A^r_n \) and \( A = \limsup_{r} A^r \). Since \( P(A^r) = 1, \forall r \in \mathbb{N} \), it is, \( \sum_{r=1}^{\infty} P(A^r) = +\infty \).

Due to second part of Borel-Cantelli lemma, it follows \( P(A) = 1 \). Let \( x \in A \) then, for \( \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, \exists r > n_0 \) such that \( x \in A^r_n \). Then, for \( \forall \varepsilon > 0 \), we have

\[
\frac{|\{(i, j) \in I_r : |S_{n+i,n+j} - 1| \geq \varepsilon, x_{n+i,n+j} = 1\}|}{|\{(i, j) \in I_r : x_{n+i,n+j} = 1\}|} = 1.
\]
Hence, subsequence $S(x)$ does not converge lacunary uniformly statistically to 1. Completely analogously, almost every subsequence $S(x)$ of sequence $S_{ij}$ does not converge lacunary uniformly statistically to 0. It follows

$$P(\{x \in X : S(x) \text{ convergent lacunary uniformly statistically}\}) = 0.$$ 

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