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CHARACTERIZATION IN TERMS OF MEASURE OF LACUNARY UNIFORM STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract

In the [3] is proven that almost every, in terms of measure P_A, subsequence $S(x)$ of double sequence S converges to L in the Pringsheim's sense, if and only if sequence S uniformly statistically converges to L. In this paper, it is proven that analogue is valid and for lacunary uniformly statistical convergence. Almost every, in terms of measure P_A, subsequence $S(x)$ of double sequence S converges to L in the Pringsheim's sense, if and only if sequence S lacunary uniformly statistically converges to L.

This is not true for measure P.

Almost every, in terms of measure P, subsequence $S(x)$ of double sequence S of 0's and 1's is not almost uniformly statistically convergent, if is sequence S lacunary uniformly statistically convergent and divergent in the Pringsheim's sense.

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1. Introduction

The concept of the statistical convergence of a sequences of real numbers was introduced by Fast [10]. Furthermore, Gökhan et al. [13] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued function. Dündar and Altay [5-9] investigated the relation between I-convergence of double sequences. Fridy and Orhan [12] have studied lacunary statistical convergence of single sequences. Patterson and Savaş in [14] defined the lacunary statistical analogue for double sequences. Now, we recall that the definitions of concepts of ideal convergence and basic concepts [1, 2, 11].

The sequence S_{ij} of real numbers converges to L in the Pringsheim's sense, if for $\forall \varepsilon > 0, \exists K > 0$ such that

$$|S_{ij} - L| \leq \varepsilon, \forall i, j \geq K.$$

We write $\lim_{i, j \to \infty} S_{ij} = L.$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{nm} be the number of $(i, j) \in K$ such that $i \leq n, j \leq m$. If

$$d_2(K) = \lim_{n, m \to \infty} \frac{K_{nm}}{nm},$$

in the Pringsheim's sense then, we say that K has double natural density. Let is sequence S_{ij} of real numbers and $\varepsilon > 0$. Let

$$A(\varepsilon) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\}.$$

The sequence $S = S_{ij}$ statistically converges to $L \in \mathbb{R}$ if $d_2(A(\varepsilon)) = 0$ for $\forall \varepsilon > 0$.

We write $st - \lim S_{ij} = L$. Let is set $X \neq \emptyset$. A class I of subsets of X is said to be an *ideal in X* provided the following statements hold:

(i) $\emptyset \in I$,

(ii) $A, B \in I \Rightarrow A \cup B \in I$,

(iii) $A \in I, B \subset A \Rightarrow B \in I$.

The ideal is called *nontrivial* if $I \neq \{\emptyset\}$ and $X \in I^c$. A nontrivial ideal I is called *admissible* if it contains all the singleton sets. A nontrivial ideal I on $\mathbb{N} \times \mathbb{N}$ is called *strongly admissible* if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

A nonempty family F of subsets of a set X is called a *filter* if

(i) $\emptyset \in F^c$,

(ii) $A, B \in F \Rightarrow A \cap B \in F$,

(iii) $A \in F, A \subset B \Rightarrow B \in F$.

In this paper, the focus is put on ideal $I_u \subset 2^{\mathbb{N} \times \mathbb{N}}$ defined by: subset A belongs to the I_u if

$$\lim_{p, q \to \infty} \frac{1}{pq} |\{i < p,\ j < q : (n + i,\ m + j) \in A\}| = 0,$$

uniformly on $n, m \in \mathbb{N}$ in the Pringsheim's sense. That is subset A of the set $\mathbb{N} \times \mathbb{N}$ is uniformly density zero.

The sequence $S = S_{ij}$ uniformly statistically converges to L if for any $\varepsilon > 0$

$$\{(i,\ j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\} \in I_u.$$

That is sequence $S = S_{ij}$ uniformly statistically converges to L if $\forall \varepsilon, \varepsilon' > 0, \exists K > 0$ such that

$$\frac{1}{pq}\left|\{i < p,\ j < q : |S_{n+i, m+j} - L| \geq \varepsilon\}\right| < \varepsilon',\ \forall p,\ q \geq K,\ \forall n,\ m \in \mathbb{N}.$$

We write $Ust - \lim S_{ij} = L$.

We denote with X a set of all double sequences of 0's and 1's, i.e.,

$$X = \{x = x_{ij} : x_{ij} \in \{0, 1\},\ i,\ j \in \mathbb{N}\}.$$

Let sequence $S = S_{ij}$ and $x \in X$. Then with $S(x)$ we denote a sequence defined following way:

$$S_{ij}(x) = S_{ij},\ \text{for}\ x_{ij} = 1.$$

The mapping $x \to S(x)$ is a bijection of the set X to a set of all subsequences of the sequence S.

Then, under the Lebesgue measure on the set of all subsequences of the sequence S consider Lebesgue measure on the set X.

Let β smallest σ-algebra subsets of the set X which contains of subsets in the form of:

$$\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1,\ x_{n_2m_2} = a_2,\ \cdots,\ x_{n_km_k} = a_k\},$$

$$a_1, \cdots, a_k \in \{0, 1\},\ k \in \mathbb{N}.$$

There is a unique Lebesgue measure P on the set X for which the following applies:

$$P(\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1,\ x_{n_2m_2} = a_2,\ \cdots,\ x_{n_km_k} = a_k\}) = \frac{1}{2^k}.$$

The subsequence $S(x)$ of sequence S uniformly statistically converges to L if $\forall \varepsilon, \varepsilon' > 0, \exists K > 0$ such that for $\forall p,\ q \geq K$ and $\forall n,\ m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{i < p,\ j < q : |S_{n+i, m+j} - L| \geq \varepsilon,\ x_{n+i, m+j} = 1\}|}{|\{i < p,\ j < q : x_{n+i, m+j} = 1\}|} \leq \varepsilon'.$$

We write $Ust - \lim S_{ij}(x) = L$.

By a lacunary sequence, we mean an increasing sequence $\Theta = (k_r)$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

Let

$$I_1 = \{(i,\ j) \in \mathbb{N} \times \mathbb{N} : i,\ j \leq k_1\},$$

$$I_2 = \{(i,\ j) \in \mathbb{N} \times \mathbb{N} : i,\ j \leq k_2\} \setminus I_1, \ldots,$$

$$I_r = \{(i,\ j) \in \mathbb{N} \times \mathbb{N} : i,\ j \leq k_r\} \setminus (I_{r-1} \bigcup I_{r-2} \bigcup \cdots \bigcup I_1), \text{ for } \forall r \in \mathbb{N}.$$

The sequence S_{ij} lacunary statistically converges to L if $\forall \varepsilon > 0$, we have

$$\lim_{r \to \infty} \frac{1}{|I_r|} |\{(i,\ j) \in I_r : |S_{ij} - L| \geq \varepsilon\}| = 0.$$

We write $S_\Theta - \lim S_{ij} = L$.

Fridy proved that if $S = S_i$ sequence of real numbers and $\Theta = (k_r)$ lacunary sequence such that

$$1 < \lim\inf \frac{k_r}{k_{r-1}} \leq \lim\ \sup \frac{k_r}{k_{r-1}} < \infty.$$

Then, sequence $S = S_i$ statistically convergent if and only if it lacunary statistically convergent.

Let $S = S_{ij}$ double sequence of real numbers and $\Theta = (k_r)$ lacunary sequence of natural numbers.

A sequence $S = S_{ij}$ lacunary uniformly statistically converges to real number L if $\forall \varepsilon, \varepsilon' > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N},$ we have

$$\frac{1}{|I_r|}|\{(i, j) \in I_r : |S_{n+i, m+j} - L| \geq \varepsilon\}| \leq \varepsilon'.$$

We write $Ust_\Theta - \lim S_{ij} = L.$

The subset A of the set $\mathbb{N} \times \mathbb{N}$ is lacunary uniformly density zero if $\forall \varepsilon > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N},$ we have

$$\frac{1}{|I_r|}|\{(i, j) \in I_r : (n + i, m + j) \in A\}| \leq \varepsilon.$$

2. New Results

Not almost every, in terms of P, subsequence $S(x)$ of double sequence S is convergent to L in the Pringsheim's sense if S converges to L lacunary uniformly statistically.

Example. Let be $\Theta = (k_r)$ lacunary sequence and $A \subset \mathbb{N} \times \mathbb{N}$ lacunary uniformly density zero such that for $\forall N \in \mathbb{N}, \exists (i, j) \in A$ and $i, j \geq N.$

Let the sequence $S = (S_{ij})$ defined as

$$S_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

Then, $\forall \varepsilon > 0, \forall n, m \in \mathbb{N},$ the following is valid:

$$\frac{1}{|I_l|}|\{(i, j) \in I_l : |S_{n+i, m+j} - 1| \geq \varepsilon\}|$$

$$= \frac{1}{|I_l|}|\{(i, j) \in I_l : (n + i, m + j) \in A\}| \to 0 \text{ for } l \to \infty.$$

Respectively, $Ust_{\Theta} - \lim S_{ij} = 1.$ Let

$$B = \bigcap_{M=1}^{\infty} \bigcup_{i,\, j \ge M}^{\infty} \{x \in X : x_{ij} = 1, (i,\ j) \in A\}.$$

$$\sum_{i,\, j \ge M,\, (i,\, j) \in A} P(\{x \in X : x_{ij} = 1\}) = \sum_{i,\, j \ge M,\, (i,\, j) \in A} \frac{1}{2} = \infty.$$

Due to second part of Borel-Cantelli lemma, $P(B) = 1.$

Since subsequence $S(x)$ of S does not converge to 1 in the Pringsheim's sense if and only if $x \in B,$ it,

$$P(\{x \in X : \lim_{i,\, j \to \infty} S_{ij}(x) = 1 \text{ in the Pringsheim's sense}\}) = 0.$$

Let $A \subset \mathbb{N} \times \mathbb{N}.$ There is a unique measure P_A on X with the property:

$$P_A(\{x \in X : x_{ij} = 1\}) = \begin{cases} \frac{1}{2}, & (i,\ j) \notin A, \\ \frac{1}{2^{i+j}}, & (i,\ j) \in A, \end{cases}$$

$$P_A(\{x \in X : x_{i_1 j_1} = a_1, \cdots, x_{i_k j_k} = a_k\})$$

$$= P_A(\{x \in X : x_{i_1 j_1} = a_1\}) \cdots P_A(\{x \in X : x_{i_k j_k} = a_k\}).$$

Analogue theorem is valid: Let the sequence $S = (S_{ij})$ be divergent in the Pringsheim's sense. Then, S uniformly statistically converges to L if and only if $\exists A \subset \mathbb{N} \times \mathbb{N}$ uniformly density zero such that

$$P_A(\{x \in X : \lim_{i,\, j \to \infty} S_{ij}(x) = L \textit{ in the Pringsheim's sense}\}) = 1.$$

Theorem 2.1. *Let the sequence* $S = (S_{ij})$ *divergent in the Pringsheim's sense. Then, the sequence S lacunary uniformly statistically converges to L if and only if* $\exists A \subset \mathbb{N} \times \mathbb{N}$ *lacunary uniformly density zero such that*

$$P_A(\{x \in X : \lim_{i,\, j \to \infty} S_{ij}(x) = L \text{ in the Pringsheim's sense}\}) = 1.$$

Proof. Because of lemma the following is valid: Let is $Ust_{\Theta} - \lim S_{ij} = L$, then $\exists A \subset \mathbb{N} \times \mathbb{N}$ lacunary uniformly density zero such that the subsequence $S(y)$ of S converges to L in the Pringsheim's sense for

$$y_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

Not generalizing we can assume that L is not a point accumulation of the subsequence $S(x)$ for

$$x_{ij} = \begin{cases} 1, & (i, j) \in A, \\ 0, & (i, j) \notin A. \end{cases}$$

Hence, the subsequence $S(z)$ converges to L in the Pringsheim's sense if and only if $\exists M \in \mathbb{N}$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : z_{ij} = 1, i, j \geq M\} \bigcap A = \emptyset.$$

Let

$$B_M = \{x \in X : x_{ij} = 1, i, j \geq M, (i, j) \in A\}, \; B = \bigcap_{M=1}^{\infty} B_M.$$

Then, $\forall M \in \mathbb{N}$, is,

$$P_A(B) \leq P_A(B_M) = \sum_{i, j \geq M, (i, j) \in A} \frac{1}{2^{i+j}} \leq \sum_{i, j \geq M} \frac{1}{2^{i+j}} = \frac{1}{2^{2M-2}}.$$

Hence, $P_A(B) = 0$. Since the set B is a set of all $x \in X$ for which $S(x)$ does not converge to L in the Pringsheim's sense. It,

$$P_A(\{x \in X : \lim_{i, j \to \infty} S_{ij}(x) = L \text{ in the Pringsheim's sense}\}) = 1.$$

Let the sequence S not be lacunary uniformly statistically convergent and let $A \subset \mathbb{N} \times \mathbb{N}$ lacunary uniformly density zero. Then, due to the lemma, the subsequence $S(x)$ is divergent in the Pringsheim's sense for

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

The following cases can be presented:

(a) $\exists (n_k), (m_k), n_k \nearrow, m_k \nearrow, (n_k, m_k) \notin A, S_{n_k m_k} \geq k$ for $\forall k$,

(b) $\exists (n_k), (m_k), n_k \nearrow, m_k \nearrow, (n_k, m_k) \notin A, S_{n_k m_k} \leq -k$ for $\forall k$,

(c) $\exists (n_k^1), (m_k^1), \exists (n_k^2), (m_k^2), (n_k^1, m_k^1), (n_k^2, m_k^2) \notin A, S_{n_k^1 m_k^1} \leq \lambda < \mu \leq S_{n_k^2 m_k^2}$. It follows:

(a) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k m_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty,$

(b) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k m_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty,$

(c) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^1 m_k^1} = 1\}) = \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^2 m_k^2} = 1\}) = \infty.$

Then due to second part of Borel-Cantelli lemma the following is valid:

(a) $P_A(\{x \in X : x_{n_k m_k} = 1 \text{ for infinite } k\}) = 1,$

(b) $P_A(\{x \in X : x_{n_k m_k} = 1 \text{ for infinite } k\}) = 1,$

(c) $P_A(\{x \in X : x_{n_k^1 m_k^1} = x_{n_k^2 m_k^2} = 1 \text{ for infinite } k\}) = 1.$

It follows

$$P_A(\{x \in X : S(x) \text{ divergent in the Pringsheim's sense}\}) = 1.$$

Hence,

$$P_A(\{x \in X : S(x) \text{ convergent in the Pringsheim's sense}\}) = 0.$$

Lemma 2.2 ([4]). *Let is* $\Theta = (k_r)$ *lacunary sequence and* $S = S_{ij}$ *double sequence. Then,* $Ust_\Theta - \lim S_{ij} = L$ *if and only if* $\exists A \subset \mathbb{N} \times \mathbb{N}$ *lacunary uniformly density zero and* $\lim_{i, j \to \infty} S_{ij}(x) = L,$ *in the Pringsheim's sense, for*

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

Proof. Let is $Ust_\Theta - \lim S_{ij} = L.$ Then there is a sequence of natural numbers $(u_r)_{r=2}^{\infty}$ such that for $\forall l \geq u_r$ and $\forall n, m \in \mathbb{N},$ we have

$$\frac{1}{|I_l|}\left|\left\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \frac{1}{r}\right\}\right| \leq \frac{1}{r}.$$

Let

$$A = \bigcup_{r=2}^{\infty}\bigcup_{n,m=1}^{\infty}\left\{(n+i, m+j) : (i, j) \in \bigcup_{l=u_r}^{u_{r+1}-1} I_l, |S_{n+i,m+j} - L| \geq \frac{1}{r}\right\}.$$

We define $x \in X$ the following way:

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

For $\forall \varepsilon > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall r \geq r_0$ we have $\frac{1}{r} \leq \varepsilon.$ From the definition of the sequence $x,$ such that for $l \geq u_{r_0}$ and $\forall n, m \in \mathbb{N}$ provided that $x_{n+i, m+j} = 1,$ we have

$$|S_{n+i, m+j}(x) - L| = |S_{n+i, m+j} - L| \leq \varepsilon.$$

Hence, for $\forall i, j \geq k_{u_{r_0}}$ and $\forall i, j \in \mathbb{N}$ provided that $x_{ij} = 1,$ we have

$$|S_{ij}(x) - L| \leq \varepsilon.$$

Hence, the subsequence $S(x)$ converges to L, in the Pringsheim's sense. For $\forall l \leq u_{r_0}$ and $\forall n, m \in \mathbb{N}$ valid

$$\frac{1}{|I_l|}|\{(i, j) \in I_l : (n+i, m+j) \in A\}|$$

$$= \frac{1}{|I_l|}\left|\left\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \frac{1}{r}\right\}\right| \leq \frac{1}{r} \leq \varepsilon.$$

Hence,

$$\lim_{l\to\infty} \frac{1}{|I_l|}|\{(i, j) \in I_l : (n+i, m+j) \in A\}| = 0, \text{ uniformly for } \forall n, m \in \mathbb{N}.$$

We assume that there is a subset A of the set $\mathbb{N} \times \mathbb{N}$ such that

$$\lim_{l\to\infty} \frac{1}{|I_l|}|\{(i, j) \in I_l : (n+i, m+j) \in A\}| = 0, \quad \text{uniformly for} \quad \forall n, m \in \mathbb{N}$$

and $\lim_{i, j\to\infty} S_{ij}(x) = L$, in the Pringsheim's sense, for

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

For $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that for $\forall n, m \geq N$, we have

$$|S_{nm}(x) - L| \leq \varepsilon.$$

For $\forall l \in \mathbb{N}$ such that $k_{l-1} > N$. Then,

$$\frac{1}{|I_l|}|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \varepsilon\}|$$

$$= \frac{1}{|I_l|}|\{(i, j) \in I_l : n+i < N \vee m+j < N, |S_{n+i, m+j} - L| \geq \varepsilon\}|$$

$$+ \frac{1}{|I_l|}|\{(i, j) \in I_l : n+i, m+j \geq N, |S_{n+i, m+j} - L| \geq \varepsilon\}|$$

$$\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|}|\{(i, j) \in I_l : n+i, m+j \geq N, (n+i, m+j) \in A\}|$$

$$\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|}|\{(i, j) \in I_l : (n+i, m+j) \in A\}|.$$

Obviously, the first summation converges to zero uniformly on $n, m \in \mathbb{N}$. The second summation converges to zero uniformly on $n, m \in \mathbb{N}$ due to the assumption.

So, $Ust_{\Theta} - \lim S_{ij} = L$.

Theorem 2.3. *Let is* $\Theta = (k_r)$ *lacunary sequence. Then, almost every double sequence of* 0's *and* 1's *is not lacunary uniformly statistically convergent.*

Proof. Let

$$A_n^r = \{x \in X : x_{n+i, n+j} = 1, (i, j) \in I_r\}.$$

Since $P(A_n^r) = \frac{1}{2^{|I_r|}}$, $\forall n \in \mathbb{N}$, it is

$$\sum_{n=1}^{\infty} P(A_n^r) = \sum_{n=1}^{\infty} \frac{1}{2^{|I_r|}} = +\infty.$$

Since A_n^k are independent, based on the second part of Borel-Cantelli lemma:

$$P\left(\limsup_n A_n^r\right) = 1.$$

We denote $A^r = \limsup_n A_n^r$, $A = \limsup_r A^r$. Since $P(A^r) = 1$, $\forall r \in \mathbb{N}$, it is $\sum_{r=1}^{\infty} P(A^r) = +\infty$.

Due to second part of Borel-Cantelli lemma, it follows $P(A) = 1$. Let is $x \in A$ then, for $\forall r_0 \in \mathbb{N}, \exists n \in \mathbb{N}, \exists r > r_0$ such that $x \in A_n^r$. It follows $\forall x \in A$ does not converge lacunary uniformly statistically to 0.

Completely analogously, almost every $x \in X$ does not converge lacunary uniformly statistically to 1.

Every lacunary uniformly statistically convergent sequence $x \in X$ converges 0 or 1. It follows

$$P(\{x \in X : x = (x_{ij}) \text{ convergent lacunary uniformly statistically}\}) = 0.$$

Definition 2.4. The subsequence $S(x)$ of sequence S lacunary uniformly statistically converges to L if $\forall \varepsilon, \varepsilon' > 0, \exists K > 0$ such that for $\forall r \geq K$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{(i, j) \in I_r : |S_{n+i, m+j} - L| \geq \varepsilon, x_{n+i, m+j} = 1\}|}{|\{(i, j) \in I_r : x_{n+i, m+j} = 1\}|} \leq \varepsilon'.$$

We write $Ust_\Theta - \lim S_{ij}(x) = L$.

Almost every, in terms of measure P, subsequence of uniformly statistically convergent double sequence S is uniformly statistically convergent. This analogue is valid also for lacunary uniform statistical convergence double sequences of 0's and 1's.

Theorem 2.5. *Let is* S_{ij} *sequence of* 0'*s and* 1'*s that is convergent lacunary uniformly statically and divergent in the Pringsheim's sense. Let is* $\Theta = (k_r)$ *lacunary sequence. Then,*

$$P(\{x \in X : S_{ij}(x) \textit{ convergent lacunary uniformly statistically}\}) = 0.$$

Proof. Since a sequence S_{ij} convergent lacunary uniformly statically, it is,

$$Ust_\Theta - \lim S_{ij} = 1 \quad \text{or} \quad 0.$$

Suppose it is $Ust_{\Theta} - \lim S_{ij} = 1$ and $\lim\limits_{i,\, j \to \infty} S_{ij} \neq 1$ in the Pringsheim's sense. Then, there exists infinite subsequence (k_{r_l}) of the sequence (k_r) such that for $\forall l \in \mathbb{N}, \exists (i,\, j) \in I_{r_l}$ we have $S_{ij} = 0$. Not generalizing we can assume that for $\forall r \in \mathbb{N}, \exists (i,\, j) \in I_r,$ we have $S_{ij} = 0$. Let

$$A_n^r = \{x \in X : x_{n+i,\, n+j} = S'_{n+i,\, n+j}, (i,\, j) \in I_r\},$$

where is

$$S'_{ij} = \begin{cases} 1, & S_{ij} = 0, \\ 0, & S_{ij} = 1. \end{cases}$$

Since $P(A_n^r) = \dfrac{1}{2^{|I_r|}}$, it is

$$\sum_{n=1}^{\infty} P(A_n^r) = \sum_{n=1}^{\infty} \frac{1}{2^{|I_r|}} = +\infty.$$

Since A_n^k are independent, based on the second part of Borel-Cantelli lemma:

$$P\left(\limsup_n A_n^r\right) = 1.$$

We denote $A^r = \limsup\limits_n A_n^r$, $A = \limsup\limits_r A^r$. Since $P(A^r) = 1, \forall r \in \mathbb{N},$ it is, $\sum_{r=1}^{\infty} P(A^r) = +\infty.$

Due to second part of Borel-Cantelli lemma, it follows $P(A) = 1$. Let is $x \in A$ then, for $\forall r_0 \in \mathbb{N}, \exists n \in \mathbb{N}, \exists r > r_0$ such that $x \in A_n^r$. Then, for $\forall \varepsilon > 0,$ we have

$$\frac{|\{(i,\, j) \in I_r : |S_{n+i,\, n+j} - 1| \geq \varepsilon,\, x_{n+i,\, n+j} = 1\}|}{|\{(i,\, j) \in I_r : x_{n+i,\, n+j} = 1\}|} = 1.$$

Hence, subsequence $S(x)$ does not converge lacunary uniformly statistically to 1. Completely analogously, almost every subsequence $S(x)$ of sequence S_{ij} does not converge lacunary uniformly statistically to 0. It follows

$$P(\{x \in X : S(x) \text{ convergent lacunary uniformly statistically}\}) = 0.$$

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