Research and Communications in Mathematics and Mathematical Sciences
Vol. 9, Issue 1, 2017, Pages 31-36
ISSN 2319-6939
Published Online on October 22, 2017

http://jyotiacademicpress.org

A FUGLEDE-PUTNAM TYPE THEOREM FOR ALMOST NORMAL OPERATORS WITH FINITE k_1-FUNCTION

VASILE LAURIC

Department of Mathematics
Florida A&M University
Tallahassee
FL 32307
USA
e-mail: vasile.lauric@famu.edu

Abstract

In this note, we will prove that operators $T \in L(\mathcal{H})$ with finite k_1 function satisfy a Fuglede-Putnam type modulo the Hilbert-Schmidt class, that is, for arbitrary $X \in L(\mathcal{H})$ with $TX - XT \in \mathcal{C}_2(\mathcal{H})$ implies $T^*X - XT^* \in \mathcal{C}_2(\mathcal{H})$.

1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_p(\mathcal{H})$ (or simply $\mathcal{C}_p$) the Shatten-von Neumann p-classes and by $|\cdot|_p$, $p \geq 1$, their respective norm. In this note only the particular classes corresponding to $p = 1, 2$ will be used, that is the trace-class $\mathcal{C}_1$ and the

2010 Mathematics Subject Classification: Primary 47B20, 47B37.
Keywords and phrases: almost normal operators, k_1-function, Fuglede-Putnam type theorem.
Received August 31, 2017

class of Hilbert-Schmidt operators $\mathcal{C}_2$. For arbitrary operators $S, T \in L(\mathcal{H})$, $[S, T]$ will denote their commutator $ST - TS$ and D_S will denote the self-commutator of S, that is $[S^*, S]$. An operator $S \in L(\mathcal{H})$ is called *almost normal* when $D_S \in \mathcal{C}_1(\mathcal{H})$ and the class of operators defined on $\mathcal{H}$ which are almost normal will be denoted by $\mathcal{AN}(\mathcal{H})$.

2. Voiculescu's Conjecture 4 (C_4), (cf. [3] or [4]) states that for $T \in AN(\mathcal{H})$, there exists $S \in AN(\mathcal{H})$ such that $T \oplus S = N + K$, where N is a normal operator and K is a Hilbert-Schmidt operator. Under the assumption that conjecture (C_4) has a positive answer, one can easily prove that almost normal operators satisfy a Fuglede-Putnam type theorem, that is, if $T \in \mathcal{AN}(\mathcal{H})$ and $X \in L(\mathcal{H})$ is such that $[T, X] \in \mathcal{C}_2(\mathcal{H})$, then $[T^*, X] \in \mathcal{C}_2(\mathcal{H})$, and the family of operators on $\mathcal{H}$ that have such a property will be denoted by $\mathcal{F}P_2(\mathcal{H})$. This results from a theorem of Weiss [5] that states that if $N \in L(\mathcal{H})$ is a normal operator and $X \in L(\mathcal{H})$ such that $[N, X] \in \mathcal{C}_2(\mathcal{H})$, then $[N^*, X] \in \mathcal{C}_2 \ (\mathcal{H})$ and $\|[N, X]\|_2 = \|[N^*, X]\|_2$ (in particular $N \in \mathcal{F}P_2(\mathcal{H})$), and the details are left for the reader.

Let $\mathcal{P}$ and $\mathcal{R}_1^+$ denote the set of finite rank orthogonal projections and the finite rank positive semidefinite contractions, respectively, and

$$q_p(T) = \liminf_{P \in \mathcal{P}} \|((I - P)TP)\|_p,$$

$$k_p(T) = \liminf_{A \in \mathcal{R}_1^+} \|[T, A]\|_p,$$

where the lim infs are with respect to the natural order.

In [1], it was proved that almost normal operators T such that $q_2(T) < \infty$ belong to $\mathcal{F}P_2(\mathcal{H})$.

3. It is natural to ask whether almost normal operators T with $k_2(T)$ finite, and implicitly $k_2(T) = 0$ (cf. [2]), belong to $\mathcal{FP}_2$.

We will prove that such a result holds under the hypothesis that $k_1(T)$ is finite.

Theorem 1. *If* $T \in \mathcal{AN}(\mathcal{H})$ *and* $k_1(T) < \infty$, *then* $T \in \mathcal{FP}_2(\mathcal{H})$.

Proof. Let $T \in \mathcal{AN}(\mathcal{H})$ with $k_1(T) < \infty$, let $A_n \in \mathcal{R}_1^+$, $n \geq 1$, so that $A_n \uparrow I$ and $|[A_n, T]|_1 \downarrow k_1(T)$, and let $X \in L(\mathcal{H})$ with $[T, X] =: R \in \mathcal{C}_2(\mathcal{H})$.

It will be enough to prove that

$$\limsup_{n\to\infty} |\mathrm{tr}[A_n(QQ^* - RR^*)]| < \infty,$$

where $Q := T^*X - XT^*$.

Write $A_nRR^* = A_nTXX^*T^* - A_nTXT^*X^* - A_nXTX^*T^* + A_nXTT^*X^* = a - b - c + d$ and $A_nQQ^* = A_nT^*XX^*T - A_nT^*XTX^* - A_nXT^*X^*T + A_nXT^*TX^* = A - B - C + D$, where $a, b, \ldots, C, D$ are the terms in the order they appear in these expansions.

First

$$|\mathrm{tr}(D - d)| \leq |D - d|_1 \leq \|X\|^2 |D_T|_1. \tag{1}$$

Then

$$|\mathrm{tr}(B - c)| = |\mathrm{tr}(A_nT^*XTX^* - A_nXTX^*T^*)|$$

$$= |\mathrm{tr}(A_nT^*XTX^* - T^*A_nXTX^*)| = |\mathrm{tr}([A_n, T^*]XTX^*)|$$

$$\leq |[A_n, T^*]XTX^*|_1 \leq |[A_n, T^*]|_1 \|XTX^*\| \leq |[A_n, T^*]|_1 \|X\|^2 \|T\|,$$

and then after passing to limit

$$|\mathrm{tr}(B - c)| \le k_1(T)\|X\|^2\|T\|. \tag{2}$$

In a similar way,

$$|\mathrm{tr}(C - b)| = |\mathrm{tr}(A_n XT^*X^*T - A_n TXT^*X^*)|$$

$$= |\mathrm{tr}(TA_n XT^*X^* - T^*A_n TXT^*X^*)| = |\mathrm{tr}([T, A_n]XT^*X^*)|$$

$$\le |[A_n, T]XT^*X^*|_1 \le |[A_n, T]|_1\|XT^*X^*\| \le |[A_n, T]|_1\|X\|^2\|T\|,$$

and thus

$$|\mathrm{tr}(C - b)| \le k_1(T)\|X\|^2\|T\|. \tag{3}$$

Finally,

$$|\mathrm{tr}(A - a)| = |\mathrm{tr}(A_n T^*XX^*T - A_n TXX^*T^*)|$$

$$= |\mathrm{tr}(TA_n T^*XX^* - T^*A_n TXX^*)| \le |TA_n T^* - T^*A_n T|_1\|X\|^2.$$

Furthermore,

$$|TA_nT^* - T^*A_nT|_1 = |TA_nT^* - A_nTT^* - A_nT^*T + A_nT^*T - T^*A_nT|_1$$

$$\le |[[T, A_n]T^*|_1 + |A_nD_T|_1 + |[A_n, T^*]T|_1$$

$$\le |[T, A_n]|_1\|T^*\| + |D_T|_1 + |[A_n, T^*]|_1\|T\|$$

$$= 2|[T, A_n]|_1\|T\| + |D_T|_1,$$

and consequently, by passing to limit, we have

$$|\mathrm{tr}(A - a)| \le (2k_1(T)\|T\| + |D_T|_1)\|X\|^2. \tag{4}$$

Using inequalities (1)-(4),

$$\limsup_{n\to\infty} |\mathrm{tr}[A_n(QQ^* - RR^*)]| \le 4k_1(T)\,\|X\|^2\|T\| + 2|D_T|_1\|X\|^2,$$

which ends the proof.

The above proof leads to the following.

Corollary 2. *If* $T \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T) < \infty$ *and* $X \in L(\mathcal{H})$ *so that* $[T, X] \in \mathcal{C}_2(\mathcal{H})$, *then* $|[T^*X]\|_2^2 \le \|[T, X]\|_2^2 + 4k_1(T)\|X\|^2\|T\| + 2|D_T|_1\|X\|^2$.

Corollary 3. *If* $T, S \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T)$ *and* $k_1(S) < \infty$ *and* $X \in L(\mathcal{H})$ *so that* $R := TX - XS \in \mathcal{C}_2(\mathcal{H})$, *then* $Q := T^*X - XS^* \in \mathcal{C}_2(\mathcal{H})$.

Proof. Let T, S, X as in the hypothesis. It is straightforward to see that

$$k_1(T \oplus S) \le k_1(T) + k_1(S) < \infty.$$

Setting $\widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, then $(T \oplus S)\widetilde{X} - \widetilde{X}(T \oplus S) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, and thus $(T \oplus S)\widetilde{X} - \widetilde{X}(T \oplus S) \in \mathcal{C}_2$. Therefore $(T \oplus S)^*\widetilde{X} - \widetilde{X}(T \oplus S)^* = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_2$. Consequently, $Q \in \mathcal{C}_2(\mathcal{H})$.

Corollary 4. *If* $T \in \mathcal{AN}(\mathcal{H})$ *with* $k_1(T) < \infty$ *and* $X \in L(\mathcal{H})$ *so that* $TX - XT^* \in \mathcal{C}_2(\mathcal{H})$, *then* $T^*X - XT \in \mathcal{C}_2(\mathcal{H})$.

References

[1] Vasile Lauric, A Fuglede-Putnam theorem modulo the Hilbert-Schmidt class for almost normal operators with finite modulus of Hilbert-Schmidt quasi-triangularity, Concr. Oper. 3 (2016), 8-14.

[2] Dan V. Voiculescu, Some results on norm-ideal perturbations of Hilbert space operators II, J. Operator Theory 5 (1981), 77-100.

[3] Dan V. Voiculescu, Hilbert space operators modulo normed ideals, Proceedings of the International Congress of Mathematicians, August 16-24, Warszawa (1983), 1041-1047.

[4] Dan V. Voiculescu, Almost normal operators mod Hilbert-Schmidt and the K-theory of the algebras $E\Lambda(\Omega)$, J. Noncommut. Geom. 8 (2014), 11231-1145.

[5] Gary Weiss, The Fuglede commutativity theorem modulo operator ideals, Proc. Amer. Math. Soc. 83 (1981), 113-118.

■