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λ-CENTRAL BMO ESTIMATES FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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Abstract

In this paper, we establish λ-central BMO estimates for the multilinear commutator related to the singular integral operator with variable Calderón-Zygmund kernel in central Morrey spaces.

1. Introduction

In recent years, research for singular integral operator is becoming more and more popular, and their commutators and multilinear operators have also been well studied (see [3-10], [12-15]). Let

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$b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f) = bT(f) - T(bf).$$

In [3] [12], the authors proved that the commutators and multilinear operators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Since $BMO \subset \bigcap_{q>1} CBMO^q$ (see [7]), if we only assume $b \in CBMO^q$, or more generally $b \in CBMO^{q,\lambda}$ with $q > 1$, then $[b, T]$ may not be a bounded operator on $L^p(R^n)$. However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos et al. ([5]) considered the commutator with $b \in CBMO^q$ on Herz spaces for the first time. Later, Alvarez et al. ([2]) and Komori ([7]) have obtained the λ-central BMO estimates for the commutators of a class of singular integral operators on central Morrey spaces. Inspired by these results, in this paper, we will establish λ-central BMO estimates for the multilinear commutator associated to the singular integral operator with variable Calderón-Zygmund kernel in central Morrey spaces.

2. Notations and Results

Definition 1. Let $0 < \lambda < 1$ and $1 < q < \infty$. A function $f \in L^q_{\text{loc}}(R^n)$ is said to belong to the λ-central bounded mean oscillation space $CBMO^{q,\lambda}(R^n)$, if

$$\|f\|_{CBMO^{q,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(0, r)|^{1+\lambda q}} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty, \quad (1)$$

where $B = B(0, r) = \{x \in R^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0, r)$.

Remark 1. If two functions which differ by a constant are regarded as a function in the space $CBMO^{q,\lambda}$ becomes a Banach space. The space $CBMO^{q,\lambda}(R^n)$ when $\lambda = 0$ is just the space $CBMO(R^n)$ defined as follows:

$$\|f\|_{CBMO_q} = \sup_{r>0}\left(\frac{1}{|B(0, r)|}\int_{B(0,r)}|f(x) - f_{B(0,r)}|^q dx\right)^{1/q} < \infty.$$

Apparently, (1) is equivalent to the following condition (see [2]):

$$\|f\|_{CBMO^{q,\lambda}} = \sup_{r>0}\inf_{c\in\mathbf{C}}\left(\frac{1}{|B(0, r)|^{1+\lambda q}}\int_{B(0,r)}|f(x) - c|^q dx\right)^{1/q} < \infty.$$

Definition 2. Let $\lambda \in \mathbf{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(R^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}} = \sup_{r>0}\left(\frac{1}{|B(0, r)|^{1+\lambda q}}\int_{B(0,r)}|f(x)|^q dx\right)^{1/q} < \infty. \tag{2}$$

Remark 2. It follows from (1) and (2) that $\dot{B}^{q,\lambda}(R^n)$ is a Banach space continuously included in $CBMO^{q,\lambda}(R^n)$. We denote by $CMO^{q,\lambda}(R^n)$ and $B^{q,\lambda}(R^n)$ the inhomogeneous versions of the λ-central bounded mean oscillation space and the central Morrey space by taking the supremum over $r \geq 1$ in Definition 1 and Definition 2 instead of $r > 0$ there. Obviously, $CBMO^{q,\lambda}(R^n) \subset CMO^{q,\lambda}(R^n)$ for $\lambda < \delta/n$ and $1 < q < \infty$, and $\dot{B}^{q,\lambda}(R^n) \subset B^{q,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $1 < q < \infty$.

Remark 3. When $\lambda_1 < \lambda_2$, it follows from the property of monotone functions that $B^{q,\lambda_1}(R^n) \subset B^{q,\lambda_2}(R^n)$ and $CMO^{q,\lambda_1}(R^n) \subset CMO^{q,\lambda_2}(R^n)$ for $1 < q < \infty$. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda}(R^n) \subset \dot{B}^{q_1,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $CBMO^{q_2,\lambda} \subset CBMO^{q_1,\lambda}$, $CMO^{q_2,\lambda}(R^n) \subset CMO^{q_1,\lambda}(R^n)$ for $0 < \lambda < 1$.

In this paper, we will study some multilinear commutators as follows (see [1]):

Definition 3. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel, if

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω is homogeneous of degree zero;

(c) $\int_{\sum} \Omega(x) x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$,

where $\sum = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 4. Let $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel, if

(d) $K(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;

(e) $\max_{|\gamma| \le 2n} \left\| \frac{\partial^{|\gamma|}}{\partial^\gamma y} \Omega(x, y) \right\|_{L^\infty(R^n \times \sum)} = M < \infty$.

Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let T be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where $K(x, x - y) = \frac{\Omega(x, x - y)}{|x - y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernel. The multilinear commutator of singular integral with variable Calderón-Zygmund kernel is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x,\, x - y) f(y) dy.$$

Note that when $m = 1$, $T_{\vec{b}}$ is just the commutator of T and b, which is widely studied (see [9-16]).

For $b_j \in CBMO^{p_{j+1}, \lambda_{j+1}}(R^n)\,(j = 1, \cdots, m)$, set

$$\|\vec{b}\|_{CBMO^{\vec{p}, \vec{\lambda}}} = \prod_{j=1}^{m} \|b_j\|_{CBMO^{p_{j+1}, \lambda_{j+1}}}.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{CBMO^{\vec{p}, \vec{\lambda}}} = \|b_{\sigma(1)}\|_{CBMO^{p_2, \lambda_2}} \cdots \|b_{\sigma(j)}\|_{CBMO^{p_{j+1}, \lambda_{j+1}}}$.

Now we state our theorems as following:

Theorem 1. *Let* $\lambda < 0$ *and* $1 < q < \infty$, *then* T *is bounded from* $\dot{B}^{q, \lambda}(R^n)$ *to* $\dot{B}^{q, \lambda}(R^n)$.

Theorem 2. *Let* $1 < q < \infty$, $1 < p_k < \infty\,(1 \leq k \leq m + 1)$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{m+1}} \leq 1$. *Suppose* $\lambda, \lambda_1 \in \mathbf{R}$, $0 < \lambda_i < 1\,(i = 2, 3, \cdots, m + 1)$, $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_{m+1}$. *If* $b_j \in CBMO^{p_{j+1}, \lambda_{j+1}}(R^n)$ *for* $j = 1, \cdots, m$, *then* $T_{\vec{b}}$ *is bounded from* $\dot{B}^{p_1, \lambda_1}(R^n)$ *to* $\dot{B}^{q, \lambda}(R^n)$, *and the following inequality holds*:

$$\|T_{\vec{b}}(f)\|_{\dot{B}^{q, \lambda}} \leq C \|\vec{b}\|_{CBMO^{\vec{p}, \vec{\lambda}}} \|f\|_{\dot{B}^{p_1, \lambda_1}}.$$

3. Proof of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 (see [1]). *Let T be the singular integral operator as Definition* 4 *and* $1 < p < \infty$. *Then T is bounded from* $L^p(R^n)$ *to* $L^p(R^n)$.

Lemma 2. *Let* $1 < p < \infty$, $\lambda > 0$. *Suppose* $b \in CBMO^{p,\lambda}(R^n)$, *then for any* $k \geq 1$, *we have*

$$|b_{2^{k+1}B} - b_B| \leq C\|b\|_{CBMO^{p,\lambda}} k|2^{k+1}B|^{\lambda}.$$

Proof.

$$\begin{aligned}
|b_{2^{k+1}B} - b_B| &\leq \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^j B}| \\
&\leq \sum_{j=0}^{k} \frac{1}{|2^j B|} \int_{2^j B} |b(y) - b_{2^{j+1}B}| dy \\
&\leq C \sum_{j=0}^{k} \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p dy \right)^{1/p} \\
&\leq C\|b\|_{CBMO^{p,\lambda}} \sum_{j=0}^{k} |2^{j+1}B|^{\lambda} \\
&\leq C\|b\|_{CBMO^{p,\lambda}} (k+1)|2^{k+1}B|^{\lambda} \\
&\leq C\|b\|_{CBMO^{p,\lambda}} k|2^{k+1}B|^{\lambda}.
\end{aligned}$$

Proof of Theorem 1. Let f be a function in $\dot{B}^{q,\lambda}(R^n)$. For fixed $r > 0$, set $B = B(0, r)$, we write

$$\left(\frac{1}{|B|^{1+\lambda q}}\int_B |T(f)(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{|B|^{1+\lambda q}}\int_B |T(f\chi_{2B})(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\frac{1}{|B|^{1+\lambda q}}\int_B |T(f\chi_{(2B)^c})(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$= I_1 + I_2.$$

For I_1, by the boundedness of T, we have

$$I_1 \leq C|B|^{-\frac{1}{q}-\lambda}\left(\int_{2B}|f(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}|B|^{\frac{1}{q}+\lambda}\|f\|_{\dot{B}^{q,\lambda}}$$

$$\leq C\|f\|_{\dot{B}^{q,\lambda}}.$$

For I_2, given $x \in B$, by Hölder's inequality, we get

$$|T(f\chi_{(2B)^c})(x)| \leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|K(x, x-y)||f(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B}|f(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\left(\int_{2^{k+1}B}|f(y)|^q\,dy\right)^{\frac{1}{q}}|2^{k+1}B|^{1-\frac{1}{q}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}|2^{k+1}B|^{\frac{1}{q}+\lambda}\|f\|_{\dot{B}^{q,\lambda}}|2^{k+1}B|^{1-\frac{1}{q}}$$

$$\leq C\|f\|_{\dot{B}^{q,\lambda}}\sum_{k=1}^{\infty}|2^kB|^{\lambda}$$

$$\leq C\|f\|_{\dot{B}^{q,\lambda}}|B|^{\lambda},$$

therefore,

$$I_2 \le C|B|^{-\frac{1}{q}-\lambda}\|f\|_{\dot{B}^{q,\lambda}}|B|^{\lambda}|B|^{\frac{1}{q}} \le C\|f\|_{\dot{B}^{p,\lambda}}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let f be a function in $\dot{B}^{p_1,\lambda_1}(R^n)$, we will consider the cases $m = 1$ and $m > 1$, respectively.

We first consider the case $m = 1$: Set $(b_1)_B = \frac{1}{|B|}\int_B b_1(y)dy$, we have

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_B)T(f)(x) - T((b_1(y) - (b_1)_B)f)(x).$$

So,

$$\left(\frac{1}{|B|^{1+\lambda q}}\int_B |T_{b_1}(f)(x)|^q dx\right)^{\frac{1}{q}}$$

$$\le \left(\frac{1}{|B|^{1+\lambda q}}\int_B |(b_1(x) - (b_1)_B)(T(f\chi_{2B}))(x)|^q dx\right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B |(b_1(x) - (b_1)_B)(T(f\chi_{(2B)^c}))(x)|^q dx\right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B |T((b_1 - (b_1)_B)f\chi_{2B})(x)|^q dx\right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B |T((b_1 - (b_1)_B)f\chi_{(2B)^c})(x)|^q dx\right)^{\frac{1}{q}}$$

$$= J_1 + J_2 + J_3 + J_4.$$

For J_1, by Hölder's inequality and boundedness of T from $L^{p_1}(R^n)$ to $L^{p_1}(R^n)$, we have

$$J_1 \le |B|^{-\frac{1}{q}-\lambda}\left(\int_B |b_1(x)-(b_1)_B|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B |T(f\chi_{2B})(x)|^{p_1}\right)^{\frac{1}{p_1}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda}|B|^{\frac{1}{p_2}+\lambda_2}\|b_1\|_{CBMO^{p_2,\lambda_2}}\left(\int_B |f(x)|^{p_1}\,dx\right)^{\frac{1}{p_1}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda+\frac{1}{p_2}+\lambda_2}\|b_1\|_{CBMO^{p_2,\lambda_2}}|B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\le C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For J_2, with the same method which we use above, we get

$$|T(f\chi_{(2B)^c})(x)| \le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|K(x,\, x-y)|\,|f(y)|\,dy$$

$$\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\left(\int_{2^{k+1}B}|f(y)|^{p_1}\,dy\right)^{\frac{1}{p_1}}|2^{k+1}B|^{1-\frac{1}{p_1}}$$

$$\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}|2^{k+1}B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}|2^{k+1}B|^{1-\frac{1}{p_1}}$$

$$\le C\|f\|_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}|2^kB|^{\lambda_1}$$

$$\le C\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1},$$

then, we can get

$$J_2 \le C|B|^{-\frac{1}{q}-\lambda}\left(\int_B |(b_1(x)-(b_1)_B)(T(f\chi_{(2B)^c}))(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}\left(\int_B |b_1(x)-(b_1)_B|^{p_2}\,dx\right)^{1/p_2}|B|^{\frac{1}{q}-\frac{1}{p_2}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda+\lambda_1+(\frac{1}{q}-\frac{1}{p_2})+(\frac{1}{p_2}+\lambda_2)}\|f\|_{\dot{B}^{p_1,\lambda_1}}\|b_1\|_{CBMO^{p_2,\lambda_2}}$$

$$\leq C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For J_3, using the boundedness of T and Hölder's inequality, we have

$$J_3 \leq C|B|^{-\frac{1}{q}-\lambda}\left(\int_B |b_1(x)-(b_1)_B f(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\left(\int_B |b_1(x)-(b_1)_B|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B |f(x)|^{p_1}\,dx\right)^{\frac{1}{p_1}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}|B|^{\frac{1}{p_2}+\lambda_2}\|b_1\|_{CBMO^{p_2,\lambda_2}}\||B|^{\frac{1}{p_1}+\lambda_1}|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\leq C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For J_4, given $x \in B$, by Hölder's inequality and Lemma 2, we have

$$|T((b_1-(b_1)_B)f\chi_{(2B)^c})(x)|$$

$$\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B}|b_1(y)-(b_1)_B||K(x,\,x-y)||f(y)|\,dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^k B|}\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_B|^{p_2}\,dy\right)^{\frac{1}{p_2}}$$

$$\times\left(\int_{2^{k+1}B}|f(y)|^{p_1}\,dy\right)^{\frac{1}{p_1}}|2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{p_2}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^k B|}\left[\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}|^{p_2}\,dy\right)^{\frac{1}{p_2}}\right.$$

$$\left.+|(b_1)_{2^{k+1}B}-(b_1)_B||2^{k+1}B|^{\frac{1}{p_2}}\right]|2^{k+1}B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}|2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{p_2}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^k B|}\left[|2^{k+1}B|^{\frac{1}{p_2}+\lambda_2}\|b_1\|_{CBMO^{p_2,\lambda_2}}\right.$$

$$+ k|2^{k+1}B|^{\lambda_2}\|b_1\|_{CBMO^{p_2,\lambda_2}}|2^{k+1}B|^{\frac{1}{p_2}}\Big]$$

$$\times |2^{k+1}B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}|2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{p_2}}$$

$$\leq C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k|2^kB|^{\lambda_1+\lambda_2}$$

$$\leq C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2},$$

therefore,

$$J_4 \leq C|B|^{-\frac{1}{q}-\lambda}\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2}|B|^{\frac{1}{q}}$$

$$\leq C\|b_1\|_{CBMO^{p_2,\lambda_2}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

This completes the proof of the case $m = 1$.

Now, we consider the case $m > 1$. Set $\vec{b}_B = ((b_1)_B, \cdots, (b_m)_B)$, where $(b_j)_B = \frac{1}{\mu(B)}\int_B |b_j(y)|d|y|$ for $1 \leq j \leq m$, we have

$$T_{\vec{b}}(f)(x) = \int_{R^n}\prod_{j=1}^{m}(b_j(x) - b_j(y))K(x, x-y)f(y)dy$$

$$= \sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b_j(x) - (b_j)_B)_\sigma$$

$$\times \int_{R^n}(b_j(y) - (b_j)_B)_{\sigma^c}K(x, x-y)f(y)dy$$

$$= \prod_{j=1}^{m}(b_j(x) - (b_j)_B)\int_R^n K(x, x-y)f(y)dy$$

$$+ (-1)^m \int_{R^n} \prod_{j=1}^{m} (b_j(y) - (b_j)_B) K(x,\ x-y) f(y) dy$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b_j(x) - (b_j)_B)_\sigma$$

$$\times \int_{R^n} (b_j(y) - (b_j)_B)_{\sigma^c} K(x,\ x-y) f(y) dy$$

$$= \prod_{j=1}^{m} (b_j(x) - (b_j)_B) T(f)(x) + (-1)^m T(\prod_{j=1}^{m} (b_j - (b_j)_B) f)(x)$$

$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b_j(x) - (b_j)_B)_\sigma T(b_j - (b_j)_B)_{\sigma^c}(f)(x),$$

thus,

$$\left(\frac{1}{|B|^{1+\lambda q}} \int_B |T_{\vec{b}}(f)(x)|^q dx \right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{|B|^{1+\lambda q}} \int_B |(b_1(x) - (b_1)_B) \cdots (b_m(x) - (b_m)_B)(T(f\chi_{2B}))(x)|^q dx \right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}} \int_B |(b_1(x) - (b_1)_B) \cdots (b_m(x) - (b_m)_B)(T(f\chi_{(2B)^c}))(x)|^q dx \right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}} \int_B |T((b_1 - (b_1)_B) \cdots (b_m - (b_m)_B) f\chi_{2B})(x)|^q dx \right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{|B|^{1+\lambda q}} \int_B |T((b_1 - (b_1)_B) \cdots (b_m - (b_m)_B) f\chi_{(2B)^c})(x)|^q dx \right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{|B|^{1+\lambda q}}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b_j(x)-(b_j)_B)_\sigma T((b_j-(b_j)_B)_{\sigma^c}f\chi_{2B})(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{|B|^{1+\lambda q}}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b_j(x)-(b_j)_B)_\sigma T((b_j-(b_j)_B)_{\sigma^c}f\chi_{(2B)^c})(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$= B_1 + B_2 + B_3 + B_4 + B_5 + B_6.$$

For B_1, by Hölder's inequality and the boundedness of T, we have

$$B_1 \le |B|^{-\frac{1}{q}-\lambda}\prod_{j=1}^m\left(\int_B|b_j(x)-(b_j)_B|^{p_{j+1}}dx\right)^{\frac{1}{p_{j+1}}}\left(\int_B|(T(f\chi_{2B}))(x)|^{p_1}dx\right)^{\frac{1}{p_1}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda}\prod_{j=1}^m\left(|B|^{\frac{1}{p_{j+1}}+\lambda_{j+1}}\|b_j\|_{CBMO^{p_{j+1},\lambda_{j+1}}}\right)\left(\int_B|f(x)|^{p_1}dx\right)^{\frac{1}{p_1}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda}|B|^{\frac{1}{p_2}+\cdots+\frac{1}{p_{m+1}}+\lambda_2+\cdots+\lambda_{m+1}}\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}|B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\le C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For B_2, by the inequality $|T(f\chi_{(2B)^c})(x)| \le C\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}$ from the proof of Theorem 1, we can get

$$B_2 \le C|B|^{-\frac{1}{q}-\lambda}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}\left(\int_B\left|\prod_{j=1}^m(b_j(x)-(b_j)_B)\right|^q dx\right)^{\frac{1}{q}}$$

$$\le C|B|^{-\frac{1}{q}-\lambda}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}\prod_{j=1}^m\left(\int_B|(b_j(x)-(b_j)_B)|^{p_{j+1}}dx\right)^{\frac{1}{p_{j+1}}}|B|^{\frac{1}{q}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}\prod_{j=1}^{m}|B|^{\frac{1}{p_{j+1}}+\lambda_{j+1}}\|b_j\|_{CBMO^{p_{j+1},\lambda_{j+1}}}|B|^{\frac{1}{q}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}$$

$$\leq C\prod_{j=1}^{m}\|b_j\|_{CBMO^{p_{j+1},\lambda_{j+1}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\leq C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For B_3, using the boundedness of T and Hölder's inequality, we have

$$B_3 \leq C|B|^{-\frac{1}{q}-\lambda}\left(\int_{2B}|(b_1(x)-(b_1)_B)\cdots(b_m(x)-(b_m)_B)f\chi_{2B}(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}\left(\int_{2B}|(b_j(x)-(b_j)_B)|^{p_{i+1}}\,dx\right)^{\frac{1}{p_{i+1}}}\left(\int_{2B}|f(x)|^{p_1}\,dx\right)^{\frac{1}{p_1}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}|2B|^{\frac{1}{p_{i+1}}+\lambda_{i+1}}\|b_i\|_{CBMO^{p_{i+1},\lambda_{i+1}}}|2B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1}$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_{CBMO^{p_{i+1},\lambda_{i+1}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\leq C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For B_4, given $x \in B$, for $\lambda, \lambda_1 \in \mathbf{R}$ and $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_{m+1}$, by Hölder's inequality and Lemma 2, we have

$$|T((b_1-(b_1)_B)\cdots(b_m-(b_m)_B)f\chi_{(2B)^c})(x)|$$

$$\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|b_1(y)-(b_1)_B|\cdots|b_m(y)-(b_m)_B\|K(x,\,x-y)\|f(y)|\,dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\prod_{j=1}^{m}\left(\int_{2^{k+1}B}|(b_j(y)-(b_j)_B)|^{p_{j+1}}\,dy\right)^{\frac{1}{p_{j+1}}}$$

$$\times\left(\int_{2^{k+1}B}|f(y)|^{p_1}\,dy\right)^{\frac{1}{p_1}}|2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}$$

$$\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\prod_{j=1}^{m}$$

$$\left[\left(\int_{2^{k+1}B}|b_j(y)-(b_j)_{2^{k+1}B}|^{p_{j+1}}\,dy\right)^{\frac{1}{p_{j+1}}}+|(b_j)_{2^{k+1}B}-(b_j)_B||2^{k+1}B|^{\frac{1}{p_{j+1}}}\right]$$

$$\times|2^{k+1}B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}|2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}$$

$$\le C\|f\|_{\dot{B}^{p_1,\lambda_1}}\prod_{j=1}^{m}\|b_j\|_{CBMO^{p_{j+1},\lambda_{j+1}}}\sum_{k=1}^{\infty}k^m|2^{k+1}B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}}$$

$$\le C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}}$$

$$= C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda},$$

so, we obtain

$$J_4\le|B|^{-\frac{1}{q}-\lambda}\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda}|B|^{\frac{1}{q}}$$

$$\le C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For B_5, let $1<q_1,\,q_2,\,q_3<\infty$, set $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$ and $\frac{1}{q_1}=\frac{1}{q_3}+\frac{1}{p_1}$, we denote $\frac{1}{q_2}=\sum\frac{1}{p_{j+1}}$, $\lambda'=\sum\lambda_{j+1}$, where j satisfies $\sigma(j)\in\sigma$, $\frac{1}{q_3}=\sum\frac{1}{p_{j+1}}$, $\lambda''=\sum\lambda_{j+1}$, $\sigma(j)\in\sigma^c$, and $\lambda_1+\lambda''<0$, by the boundedness of T and Hölder's inequality, we have

$$B_5\le C|B|^{-\frac{1}{q}-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B|(b_j(x)-b_{jB})_\sigma|^{q_2}\,dx\right)^{\frac{1}{q_2}}$$

$$\times\left(\int_B |T((b_j - b_{j_B})_{\sigma^c} f\chi_{2B})(x)|^{q_1} dx\right)^{\frac{1}{q_1}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B |(b_j(x) - b_{j_B})_\sigma|^{q_2} dx\right)^{\frac{1}{q_2}}$$

$$\times\left(\int_B |(b_j - (b_j)_B)_{\sigma^c} f(x)|^{q_1} dx\right)^{\frac{1}{q_1}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B |(b_j(x) - b_{j_B})_\sigma|^{q_2} dx\right)^{\frac{1}{q_2}}$$

$$\times\left(\int_B |(b_j - (b_j)_B)_{\sigma^c}|^{q_3} dx\right)^{\frac{1}{q_3}}\left(\int_B |f(x)|^{p_1} dx\right)^{\frac{1}{p_1}}$$

$$\leq C|B|^{-\frac{1}{q}-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|B|^{\frac{1}{q_2}+\lambda'}\|\vec{b}_\sigma\|_{CBMO^{q_2,\lambda'}}|B|^{\frac{1}{q_3}+\lambda''}\|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}}$$

$$\times|B|^{\frac{1}{p_1}+\lambda_1}\|f\|_{\dot{B}^{p_1,\lambda_1}}$$

$$\leq C\|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}}\|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

For B_6, given $x \in B$, using the same notations in B_5, $\lambda = \lambda_1 + \lambda' + \lambda''$, by Hölder's inequality and Lemma 2, we have

$$|T((b_j - b_{j_B})_{\sigma^c} f\chi_{(2B)^c})(x)|$$

$$\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B}|(b_j - b_{j_B})_{\sigma^c}||K(x, x-y)||f(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{|2^k B|}\left(\int_{2^{k+1}B}|(b_j - b_{j_B})_{\sigma^c}|^{q_3} dy\right)^{\frac{1}{q_3}}$$

$$\times \left(\int_{2^{k+1}B} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} |2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{q_3}}$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{|2^k B|} |2^{k+1}B|^{\frac{1}{q_3}+\lambda''} \|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}} |2^{k+1}B|^{\frac{1}{p_1}+\lambda_1}$$

$$\times \|f\|_{\dot{B}^{p_1,\lambda_1}} |2^{k+1}B|^{1-\frac{1}{p_1}-\frac{1}{q_3}}$$

$$\leq C \|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}} \|f\|_{\dot{B}^{p_1,\lambda_1}} \sum_{k=1}^{\infty} |2^k B|^{\lambda_1+\lambda''}$$

$$\leq C \|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}} \|f\|_{\dot{B}^{p_1,\lambda_1}} \mu(B)^{\lambda_1+\lambda''},$$

thus,

$$B_6 \leq C\mu(B)^{-\frac{1}{q}-\lambda} \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}} \|f\|_{\dot{B}^{p_1,\lambda_1}} \mu(B)^{\lambda_1+\lambda''}$$

$$\times \left(\int_B |(b_j(x) - (b_j)_B)_\sigma|^{q_2} d\mu(x) \right)^{\frac{1}{q_2}} \mu(B)^{\frac{1}{q}-\frac{1}{q_2}}$$

$$\leq C\mu(B)^{-\frac{1}{q}-\lambda} \|\vec{b}_{\sigma^c}\|_{CBMO^{q_3,\lambda''}} \mu(B)^{\lambda_1+\lambda''} \|f\|_{\dot{B}^{p_1,\lambda_1}} \|\vec{b}_\sigma\|_{CBMO^{q_2,\lambda'}}$$

$$\times \mu(B)^{\frac{1}{q_2}+\lambda'} \mu(B)^{\frac{1}{q}-\frac{1}{q_2}}$$

$$\leq C \|\vec{b}\|_{CBMO^{\vec{p},\vec{\lambda}}} \|f\|_{\dot{B}^{p_1,\lambda_1}}.$$

This completes the total proof of the Theorem 2.

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