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ON THE STABILITY OF σ QUADRATIC FUNCTIONAL EQUATION

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Abstract

In this paper, we establish the generalized Hyers-Ulam stability of the equation

$$f(ax+by) = a^2 f(x) + b^2 f(y) + \frac{ab}{2}[f(x+y) - f(x+\sigma(y))],$$

and

$$f(ax+by) = a^2 g(x) + b^2 h(y) + \frac{ab}{2}[f(x+y) - f(x+\sigma(y))]$$

on Banach spaces, by using the fixed theorem.

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1. Introduction

The problem of the stability of functional equation has originally been started by Ulam [25]. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by Aoki [1] for additive mappings and by Rassias [22] for linear mappings. The paper of Rassias [22] has been an influential in the development of what is now known as the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. A generalization of the Rasssias theorem was obtained by Gavruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \tag{1.1}$$

is called the quadratic functional equation. A generalized Hyers-Ulam stability for the quadratic functional equation was proved by Skof [24] for the function $f : X \to Y,$ where X is a normal space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. Czerwik [4] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). Park [19] proved the generalized Hyers-Ulam stability of the quadratic functional equation in Banach modules over a $\mathbb{C}^*$ algebra. The stability problem of several functional equation have been extensively investigated by number of mathematicians ([2], [13], [14], [15], [17], [18], [19], [22]).

We recall the following theorem by Diaz and Marglis: Let X be a set, a function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 ([8]). *Let* (X, d) *be a complete generalized metric space and* $J : X \to X$ *be a strict contractive mapping with a Lipschitz constant* $L \leq 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integer n or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1}x) \prec \infty \forall n \geq n_0$;

(2) *the sequence* $J^n x$ *converge to a fixed* y^* *for J*;

(3) y^* *is the unique fixed point of J in the set* $Y = \{y \in X, d(J^{n_0}x, y) \prec \infty\}$.

In [16], Najati and Park showed that the functional equation

$$f(ax + by) = a^2 f(x) + b^2 f(y) + \frac{ab}{2}[f(x + y) - f(x - y)], \tag{1.2}$$

is equivalent to the quadratic functional equation (1.1), if a, b are rational numbers such that $a^2 + b^2 \neq 1$ and, they proved the stability problem of this equation.

Throughout this paper, assume that X is a normed vector space with $\|.\|$, Y is a Banach space with norm $\|.\|$, and suppose $\sigma(\sigma(x)) = x$ for all $x \in X$. In this paper, using the fixed point theorem, we will prove the generalized stability of the following equation:

$$f(ax + by) = a^2 f(x) + b^2 f(y) + \frac{ab}{2}[f(x + y) - f(x + \sigma(y))], \tag{1.3}$$

and

$$f(ax + by) = a^2 g(x) + b^2 h(y) + \frac{ab}{2}[f(x + y) - f(x + \sigma(y))]. \tag{1.4}$$

2. Hyers-Ulam Stability of Quadratic Functional Equations

In this section, we take $f : X \to Y$ and we define

$$Df(x, y) = f(ax + by) - a^2 f(x) - b^2 f(y) - \frac{ab}{2}[f(x + y) - f(x + \sigma(y))],$$

and

$$Df_g^h(x, y) = f(ax + by) - a^2 g(x) - b^2 h(y) - \frac{ab}{2}[f(x + y) - f(x + \sigma(y))],$$

where a, b in $\mathbb{N} - \{0, 1\}$.

Theorem 2.1. *A mapping* $f : X \to Y$ *satisfies*

$$f(ax + by) = a^2 f(x) + b^2 f(y) + \frac{ab}{2}[f(x + y) - f(x + \sigma(y))], \tag{2.1}$$

if and only if f satisfies

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y) \text{ and } f(x + \sigma(x)) = 0, \tag{2.2}$$

for all $x, y \in X$.

Proof. Suppose that (2.1) holds. Since $a^2 + b^2 \neq 1$, letting $x = y = 0$ in (2.1), we get $f(0) = 0$. Letting $y = 0$ in (2.1), we obtain

$$f(ax) = a^2 f(x), \tag{2.3}$$

for all $x \in X$, and putting $x = 0$, we get

$$f(by) = b^2 f(y) + ab(f(y) - f(\sigma(y))), \tag{2.4}$$

for all $y \in Y$. Now putting $x = bx$ and $y = ay$ in (2.1), we get

$$f(abx + aby) = a^2 f(bx) + b^2 f(ay) + \frac{ab}{2}(f(bx + ay) - f(bx + a\sigma(y))), \tag{2.5}$$

for all x, y in X. Letting $y = \sigma(y)$ in (2.5), we get

$$f(abx + ab\sigma(y)) = a^2 f(bx) + b^2 f(a\sigma(y)) + \frac{ab}{2}(f(bx + a\sigma(y)) - f(bx + ay)), \tag{2.6}$$

for all x, y in X, by (2.2), (2.5), and (2.6) and we obtain

$$f(bx + by) + f(bx + b\sigma(y)) = 2f(bx) + b^2(f(y) + f(\sigma(y))). \tag{2.7}$$

Using (2.4), we get

$$f(by) + f(b\sigma(y)) = b^2(f(y) + f(\sigma(y))).$$

Then

$$f(bx + by) + f(bx + b\sigma(y)) = 2f(bx) + (f(by) + f(b\sigma(y))). \tag{2.8}$$

Then, f satisfies

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + (f(y) + f(\sigma(y))), \tag{2.9}$$

we replace x and y by $x + \sigma(x)$ in (2.9), we obtain

$$f(2(x + \sigma(x))) = 2f(x + \sigma(x)).$$

Now, we will prove that f satisfies $f(x + \sigma(x)) = 0$ for all $x \in X$. By applying the inductive argument, we show that

$$f(n(x + \sigma(x))) = nf(x + \sigma(x)),$$

for all $x \in X$ and for all $n \in \mathbb{N}$. Replacing x and y by $x + \sigma(x)$, we find $f(2(x + \sigma(x))) = 2f(x + \sigma(x))$. Writing $n(x + \sigma(x))$ instead of x and $x + \sigma(x)$ instead of y in (2.9), we get

$$f((n + 1)(x + \sigma(x))) = (n + 1)f(x + \sigma(x)).$$

This proves for all $n \in \mathbb{N}$. By using (2.4), we obtain $b^2 f(x + \sigma(x)) = bf(x + \sigma(x))$ for all $x \in X$, since $b \neq 1$ and $f(b(x + \sigma(x)) = bf(x + \sigma(x))$, then we get $f(x + \sigma(x)) = 0$, for all $x \in X$, we replace x and y by x in (2.9), we get

$$f(2x) = 3f(x) + f(\sigma(x)),$$

we use the inductive argument to prove that there exists α_n and β_n such that $\alpha_n^2 + \beta_n^2 = n^2$ and

$$f(nx) = \alpha_n f(x) + \beta_n f(\sigma(x)),$$

we have

$$f((n+1)x) + f((n-1)x + x + \sigma(x)) = 2f(nx) + f(x) + f(\sigma(x)),$$

and

$$f((n-1)x + x + \sigma(x)) = f((n-1)x).$$

Then

$$f((n+1)x) = (2\alpha_n - \alpha_{n-1} + 1)f(x) + (2\beta_n - \beta_{n-1} + 1)f(\sigma(x)),$$

this complete inductive argument. By $f(ax) = \alpha_n f(x) + \beta_n f(\sigma(x))$ and $f(ax) = a^2 f(x)$, we get $f(\sigma(x)) = f(x)$.

We shall now prove the converse. Let $f : E \to F$ be a solution of Equation (2.2). Replacing x by $(n-1)x$ and y by $x + \sigma(x)$ in (2.2), we obtain the equation $f(nx + \sigma(x)) = f((n-1)x)$, for all $x \in E$ and for all $n \in \mathbb{N}^*$. We will prove by the mathematical induction that

$$f(nx + y) = n^2 f(x) + f(y) + \frac{n}{2}(f(x+y) - f(x + \sigma(y)). \tag{2.10}$$

The result for $n = 1$ is immediately.

We prove now (2.10) is satisfied for $n + 1$. By using the hypothesis inductive, we find that

$$f((n+1)x + y) = n^2 f(x) + f(x+y) + \frac{n}{2}(f(2x+y) - f(x + \sigma(x) + \sigma(y))).$$

We have

$$f(x + \sigma(x) + \sigma(y)) = f(\sigma(y)) = f(y),$$

then we get the result. Finally, we have

$$f(nx) = n^2 f(x), \tag{2.11}$$

for all $n \in \mathbb{N}$,

$$f(ax + by) = a^2 f(x) + f(by) + \frac{a}{2}(f(x + by) - f(x + b\sigma(y))),$$

$$f(x+by) = b^2 f(y) + f(x) + \frac{b}{2}(f(x+y) - f(\sigma(x)+y)),$$

$$f(x+b\sigma(y)) = b^2 f(\sigma(y)) + f(x) + \frac{b}{2}(f(x+\sigma(y)) - f(x+\sigma(y))).$$

Since f is even, we get the result. This completes the proof of Theorem 2.1. □

Using the fixed point method, we prove the stability of the σ-quadratic functional equation $Df(x, y) = 0$.

Theorem 2.2. *Let* $f : X \to Y$ *for which there exists a function* $\phi : X^2 \to [0, \infty)$

$$\|Df(x, y)\| \leq \phi(x, y), \tag{2.12}$$

for all $x, y \in X$, *and* $\psi(x, y) = \phi(x, y) + \phi(0, 0)$ *such that*

$$\lim_{n\to+\infty} a^{-2n}\phi(a^n x, a^n y) = 0, \tag{2.13}$$

for all $x \in X$. *Let* $L \prec 1$ *such that* $\phi(x, 0) \leq a^2 L\phi(\frac{x}{a}, 0)$, *for all* $x \in X$. *Then there exists a mapping* $Q : X \to Y$ *satisfying* (2.2) *and*

$$\|f(x) - f(0) - Q(x)\| \leq \frac{1}{a^2 - a^2 L}\psi(x, 0), \tag{2.14}$$

for all $x \in X$.

Proof. Considering $F(x) = f(x) - f(0)$, the inequality (2.12) becomes $\|DF(x, y)\| \leq \psi(x, y)$. Let the set

$$S = \{g : X \to Y\}, \tag{2.15}$$

and introduce the generalized metric on S as follows:

$$d(g, h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\psi(x, 0),\ \forall x \in X\}. \tag{2.16}$$

It is easy to show that (S, d) is complete. (see the proof of Theorem 2.5 of [8]). Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{a^2} g(ax), \tag{2.17}$$

for all $x \in X$.

We have

$$\left\| \frac{g(ax)}{a^2} - \frac{h(ax)}{a^2} \right\| \leq \frac{d(g, h)\psi(ax, 0)}{a^2} \leq Ld(g, h),$$

then

$$d(Jg, Jh) \leq Ld(g, h), \tag{2.18}$$

for all $x \in S$.

Letting $y = 0$ and in (2.12), we get

$$\| f(ax) - a^2 f(x) \| \leq \psi(x, 0), \tag{2.19}$$

for all $x \in X$. So

$$d(F, JF) \leq \frac{1}{a^2}, \tag{2.20}$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q(ax) = a^2 Q(x), \tag{2.21}$$

for all $x \in X$. The Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) \leq \infty\}. \tag{2.22}$$

This implies that Q is a unique mapping (2.9) such that there exists $K \in (0, \infty)$ satisfying

$$\| F(x) - Q(x) \| \leq K\psi(x, 0), \tag{2.23}$$

for all $x \in X$.

(2)

$$\lim_{n \to +\infty} J^n F(x) = \lim_{n \to +\infty} \frac{F(a^n x)}{a^{2n}} = Q(x), \tag{2.24}$$

for all $x \in X$.

(3) $d(F, Q) \leq \frac{1}{1 - L} d(F, JF)$, which implies the inequality

$$d(F, Q) \leq \frac{1}{a^2 - a^2 L}. \tag{2.25}$$

This implies that the inequality (2.14) holds.

It follows from (2.12), (2.13), and (2.24) that

$$\|D(Q(x, y))\| = \lim_{n \to +\infty} \frac{\|DF(a^n x, a^n y)\|}{a^{2n}} \leq \lim_{n \to +\infty} \frac{\psi(a^n x, a^n y)}{a^{2n}} = 0, \tag{2.26}$$

for all $x, y \in X$. So $DQ(x, y) = 0$ for all $x, y \in X$. □

Corollary 2.3. *Let* $p \prec 2$ *and* $\theta \geq 0$ *be real numbers, and let* $f : X \to Y$ *be a mapping such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{2.27}$$

for all $x, y \in X$. *Then, there exists a unique mapping* $Q : X \to Y$ *satisfying* (2.2) *and*

$$\|f(x) - Q(x)\| \leq \frac{1}{a^2 - a^p}, \tag{2.28}$$

for all $x \in X$.

Proof. We get the result from Theorem 2.2 by taking

$$\phi(x, y) := \psi(x, y) := \theta(\|x\|^p + \|y\|^p), \tag{2.29}$$

for all $x, y \in X$. We have $f(0) = 0$, by letting $x = y = 0$ in (2.28) and we can choose $L = b^{p-2}$, then we get the desired result. □

Theorem 2.4. *Let $f, g, h : X \to Y$ be an even mapping for which there exists a function $\phi : X^2 \to [0, \infty)$ satisfying*

$$\|Df_g^h(x, y)\| \leq \phi(x, y), \tag{2.30}$$

and

$$\lim_{n \to +\infty} 2^{-2n} \phi(2^n x, 2^n y) = 0. \tag{2.31}$$

Let $\Phi(x, y) = \phi(x, y) + \phi(0, 0)$ and let

$$\psi(x, y) = \Phi\left(\frac{x}{a}, \frac{y}{b}\right) + \Phi\left(\frac{x}{a}, \frac{\sigma(y)}{b}\right) + 2\Phi\left(\frac{x}{a}, 0\right) + 2\Phi\left(0, \frac{y}{b}\right),$$

and $L \prec 1$ such that

$$\psi(x, x) \leq 4L\left(\psi\left(\frac{x}{2}, \frac{x}{2}\right)\right), \tag{2.32}$$

for all $x \in X$. Then, there exists a unique mapping $Q : X \to Y$ satisfying (2.1) *and*

$$\left\| f(x) - \frac{1}{2}(f(x + \sigma(x)) + f(0)) - Q(x) \right\| \leq \frac{1}{4(1-L)}(\psi(x, x) + \psi(x + \sigma(x), x + \sigma(x)), \tag{2.33}$$

$$\left\| g(x) - \frac{1}{2}(g(x + \sigma(x)) + g(0)) - Q(x) \right\| \leq \frac{1}{a^2}\left(\Phi(x, 0) + \frac{1}{2}(\Phi(x + \sigma(x)))\right.$$

$$\left. + \frac{1}{4(1-L)}(\psi(x, x) + \psi(x + \sigma(x), x + \sigma(x))\right),$$

and

$$\left\| h(x) - \frac{1}{2}(h(x + \sigma(x)) + h(0)) - Q(x) \right\| \leq \frac{1}{b^2}\left(\Phi(0, x) + \frac{1}{2}(\Phi(0, x + \sigma(x)))\right.$$

$$+\frac{1}{4(1-L)}(\psi(x, x)+\psi(x+\sigma(x), x+\sigma(x))),$$

for all $x \in X$.

Proof. Put $F(x) = f(x) - f(0)$, $G(x) = g(x) - g(0)$, $H(x) = h(x) - h(0)$, we have

$$\|DF_G^H(x, y) + DF_G^H(x, \sigma(y)) - 2DF_G^H(x, 0) - 2DF_G^H(0, y)\| \leq \psi(ax, by), \tag{2.34}$$

for all $x, y \in X$. Therefore,

$$\|F(ax+by) + F(ax+b\sigma(y)) - 2F(x) - 2F(y)\| \leq \psi(ax, by), \tag{2.35}$$

for all $x, y \in X$. Replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$ in (2.35), we get

$$\|F(x+y) + F(x+\sigma(y)) - 2F(x) - 2F(y)\| \leq \psi(x, y), \tag{2.36}$$

for all $x, y \in X$. Consider the set

$$S = \{l : X \to Y\},$$

and introduce the generalized metric on S as follows:

$$d(l, k) = \inf\{K \in \mathbb{R}_+ : \|l(x) - k(x)\| \leq K(\Psi(x)), \forall x \in X\},$$

with $\Psi(x) = \psi(x, x) + \frac{1}{4}\psi(x+\sigma(x), x+\sigma(x))$. It is easy to show that (S, d) is complete. (See the proof of Theorem 2.5 of [8].)

Now put $F_1(x, y) = F(x) - \frac{1}{2}F(x+\sigma(x))$, we use the inequality (2.36) and we replace x by $x+\sigma(x)$ and y by $y+\sigma(y)$ in (2.36), we get

$$\|F_1(x+y) - F_1(x+\sigma(y)) - 2F_1(x) - 2F_1(y)\| \leq \psi(x, x) + \frac{1}{4}(\psi(x+\sigma(x), y+\sigma(y)). \tag{2.37}$$

If you replace in the first x and y by x in (2.36) and in the second x and y by $x+\sigma(x)$ in (2.37), we obtain

$$\|F_1(2x)-4F_1(x)\| \leq \Psi(x). \tag{2.38}$$

Now, we consider the linear mapping $J: S \to S$ such that

$$Jk(x) := \frac{k(2x)}{4}, \tag{2.39}$$

for all $x \in X$.

Similar to the proof of Theorem 2.3, we deduce that the sequence $J^n F_1$ converges to a fixed point Q of J. Also Q is the unique fixed point of J on the set $M = \{g \in S : d(f, g) < \infty\}$, hence Q satisfies (2.2) and $Q(2x) = 4Q(x)$, then $Q(x+\sigma(x)) = 0$, so Q is solution of (2.1) and satisfying (2.33).

Now, we put $G_1(x) = G(x) - \frac{1}{2}G(x+\sigma(x))$ and $H_1(x) = H(x) - \frac{1}{2}H$ $(x+\sigma(x))$ by (2.30), we have

$$\|F_1(ax) - a^2 G_1(x)\| \leq \Phi(x, 0) + \frac{1}{2}\Phi(x+\sigma(x), 0),$$

and

$$\|F_1(ax) - b^2 H_1(x)\| \leq \Phi(0, x) + \frac{1}{2}\Phi(0, x+\sigma(x)).$$

Then, we get

$$\|Q(ax) - a^2 G_1(x)\| \leq \Phi(x, 0) + \frac{1}{2}\Phi(x+\sigma(x), 0)$$
$$+ a^2 \frac{1}{4(1-L)}(\psi(x, x) + \psi(x+\sigma(x), x+\sigma(x))),$$

and

$$\|Q(bx) - b^2 H_1(x)\| \leq \Phi(0, x) + \frac{1}{2}\Phi(0, x+\sigma(x))$$

$$+ b^2 \frac{1}{4(1-L)} (\psi(x, x) + \psi(x + \sigma(x), x + \sigma(x))),$$

which ends the proof.

Corollary 2.5. *Let* $f : X \to Y$ *be mapping and* a, b *in* $\mathbb{N}^*$, *for which there exists a function* $\phi : X^2 \to [0, \infty)$ *satisfying*

$$\|Df(x, y)\| \leq \phi(x, y), \tag{2.40}$$

and

$$\lim_{n\to+\infty} 2^{-2n} \phi(2^n x, 2^n y) = 0. \tag{2.41}$$

Let $\Phi(x, y) = \phi(x, y) + \phi(0, 0)$ *and let*

$$\psi(x, y) = \Phi(\frac{x}{a}, \frac{y}{b}) + \Phi(\frac{x}{a}, \frac{\sigma(y)}{b}) + 2\Phi(\frac{x}{a}, 0) + 2\Phi(0, \frac{y}{b}),$$

and $L \prec 1$ *such that*

$$\psi(x, x) \leq 4L(\psi(\frac{x}{2}, \frac{x}{2})), \tag{2.42}$$

for all $x \in X$. *Then, there exists a unique mapping* $Q : X \to Y$ *satisfying* (2.1) *and*

$$\left\| f(x) - \frac{1}{2}(f(x + \sigma(x)) - f(0)) - Q(x) \right\| \leq \frac{1}{4(1-L)} (\psi(x, x) + \psi(x + \sigma(x), x + \sigma(x))). \tag{2.43}$$

Corollary 2.6. *Let* $f, g, h : X \to Y$ *be an even mapping for which there exists a function* $\phi : X^2 \to [0, \infty)$ *satisfying*

$$\left\| f(ax + by) - a^2 f(x) + b^2 h(y) - \frac{ab}{2} (f(x + y) - f(x - y)) \right\| \leq \phi(x, y), \tag{2.44}$$

and

$$\lim_{n\to+\infty} 2^{-2n} \phi(2^n x, 2^n y) = 0. \tag{2.45}$$

Let $\Phi(x, y) = \phi(x, y) + \phi(0, 0)$ *and let*

$$\psi(x, y) = \Phi\left(\frac{x}{a}, \frac{y}{b}\right) + \Phi\left(\frac{x}{a}, \frac{-y}{b}\right) + 2\Phi\left(\frac{x}{a}, 0\right) + 2\Phi\left(0, \frac{y}{b}\right),$$

and $L < 1$ *such that*

$$\psi(x, x) \leq 4L\left(\psi\left(\frac{x}{2}, \frac{x}{2}\right)\right), \tag{2.46}$$

for all $x \in X$. *Then, there exists a unique mapping* $Q : X \to Y$ *satisfying* (2.1)

$$\|f(x) - f(0) - Q(x)\| \leq \frac{1}{4(1-L)}(\psi(x, x) + \psi(0, 0)), \tag{2.47}$$

$$\|g(x) - g(0) - Q(x)\| \leq \frac{1}{a^2}\Phi(x, 0) + \frac{1}{2}\Phi(0, 0) + \frac{1}{4(1-L)}(\psi(x, x) + \psi(0, 0)),$$

and

$$\|h(x) - h(0) - Q(x)\| \leq \frac{1}{b^2}\Phi(0, x) + \frac{1}{2}\Phi(0, 0) + \frac{1}{4(1-L)}(\psi(x, x) + \psi(0, 0)),$$

for all $x \in X$.

Proof. By Theorem 2.4 and $\sigma(x) = -x$, we get the result. □

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