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ON THE REGULARITY OF THE DISPLACEMENT SEQUENCE OF AN ORIENTATION PRESERVING CIRCLE HOMEOMORPHISM

WACŁAW MARZANTOWICZ[1] **and JUSTYNA SIGNERSKA**[2]

[1]Faculty of Mathematics and Computer Science
Adam Mickiewicz University of Poznań
ul. Umultowska 87
61-614 Poznań
Poland
e-mail: marzan@amu.edu.pl

[2]Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-956 Warszawa
Poland
e-mail: j.signerska@impan.pl

Abstract

We investigate the regularity properties of the displacement sequence $\eta_n(z) = \Phi^n(x) - \Phi^{n-1}(x) \bmod 1,\ z = \exp(2\pi i x)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a lift of an orientation preserving circle homeomorphism. If the rotation number $\varrho(\varphi) = \frac{p}{q}$ is rational, then $\eta_n(z)$ is asymptotically periodic with semi-period q. This

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convergence to a periodic sequence is uniform in z if we admit that some points are iterated backward instead of taking only forward iterations for all z. This leads to the notion of an ε-basins' edge, which we illustrate by the numerical example. If $\varrho(\varphi) \notin \mathbb{Q}$, then some classical results in topological dynamics yield that the displacement sequence also exhibits some regularity properties, which we define and prove in the second part of the paper.

1. Introduction

A particular example when the displacement sequence of a circle map is considered, are the so-called *interspike-intervals* for periodically driven *integrate-and-fire* models of neuron's activity (see, for example, [3, 10, 12]). In these usually one-dimensional models, a continuous dynamics induced by the differential equation is interrupted by the threshold and reset behaviour

$$\dot{x} = f(t, x), \quad f : \mathbb{R}^2 \to \mathbb{R},$$

$$\lim_{t \to s^+} x(t) = x_r \quad \text{if } x(s) = x_\Theta,$$

meaning that once a dynamical variable $x(t)$ starting at time t_0 from a resting value $x = x_r$ reaches a certain threshold x_Θ at some time t_1, it is immediately reset to a resting value and the system evolves again from a new initial condition (x_r, t_1) until some time t_2 when the threshold is reached again, etc. The question is to describe the sequence of consecutive resets t_n as iterations of some map $\Phi^n(t_0)$, called the *firing map*, and the sequence of interspike-intervals $t_n - t_{n-1}$ (time intervals between the resets) as a sequence of displacements $\Phi^n(t_0) - \Phi^{n-1}(t_0)$ along a trajectory of this map. The problem appears in various applications, such as modelling of an action potential (spiking) by a neuron, cardiac rhythms and arrhythmias ([1]) or electric discharges in electrical circuits (see [4] and references therein). Analysis of the behaviour of the displacement sequence of trajectories of an orientation preserving homeomorphism of the circle covers an answer to this question for the firing map induced by a function f regular enough and periodic in t-variable. This special type systems were, partially, our motivation for the study of the displacement sequence. In particular, the (asymptotic) periodicity of interspike-intervals is associated with the *phase-locking* phenomenon (see, e.g., [5, 8]) in integrate-and-fire models under periodic drive.

We start with homeomorphisms with rational rotation number, where, in particular, we show the connection between semi-periodic circle homeomorphism and the notion of a semi-periodic sequence and introduce the concept of an ε-basins' edge, separating the points, for which the displacement sequence becomes periodic with given ε-accuracy faster (in terms of number of iterates) when iterated forward than when iterated backward, and the points with the opposite property. In next, we consider homeomorphisms with irrational rotation number, where with the use of topological dynamics we show how the recurrent properties of points iterated under φ are reflected in the displacement sequence.

For the homeomorphism with irrational rotation number, the other interesting issue is the distribution of displacements μ_Ψ with respect to the unique ergodic probability measure μ and its behaviour under perturbation of the given homeomorphism. This question is treated in detail in the recent paper [13], which together with this work completes the description of the properties of the displacement sequence for an orientation preserving circle homeomorphism.

Let $\varphi : S^1 \to S^1$ be a map and $\Phi : \mathbb{R} \to \mathbb{R}$ be its lift, where $\mathbb{R}$ covers S^1 by the covering projection $\mathfrak{p} : x \mapsto \exp(2\pi i x)$. If $\varphi : S^1 \to S^1$ is an orientation preserving homeomorphism, then $\Phi(x+1) = \Phi(x) + 1$ for all $x \in \mathbb{R}$.

Definition 1.1. For $x \in \mathbb{R}$, the limit

$$\varrho(\Phi)(x) := \lim_{n \to \infty} \frac{\Phi^n(x)}{n}, \tag{1}$$

is called the *rotation number* of Φ at x provided the limit exists.

Remark 1.2. Let $\Psi(x) := \Phi(x) - x$ be the displacement function associated with Φ. Then

$$\varrho(\Phi)(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Psi(\Phi^{n-1}(x)). \tag{2}$$

If Φ is a lift of an orientation preserving homeomorphism $\varphi : S^1 \to S^1$, then $\varrho(\Phi)(x)$ exists and does not depend on x, following the classical Poincaré theory. In this case, we define $\varrho(\varphi) := \varrho(\Phi) \bmod 1$, where Φ is any lift of φ. Since throughout the rest of the paper, we will consider only orientation preserving circle homeomorphisms, we skip the assumption that φ preserves orientation in formulation of the forthcoming theorems and definitions.

Definition 1.3. The sequence

$$\eta_n(z) := \Psi(\Phi^{n-1}(x)) \bmod 1 = \Phi^n(x) - \Phi^{n-1}(x) \bmod 1,\ n = 1, 2, \ldots, \tag{3}$$

will be called the *displacement sequence* of a point $z = \exp(2\pi i x) \in S^1$.

Note that $\eta_n(z)$ can be seen as an arc length from the point $\varphi^{n-1}(z)$ to $\varphi^n(z)$ with respect to the positive orientation of S^1. In particular, it does not depend on a choice of the lift Φ.

At first we make two simple observations.

Remark 1.4. If φ is a rotation by $2\pi\varrho$, where ϱ can be either rational or irrational, then the sequence $\eta_n(z)$ is constant (as the rotation is an isometry). Precisely, $\eta_n(z) = \varrho$ for all $z \in S^1$ and $n \in \mathbb{N}$.

Remark 1.5. If φ is conjugated to the rational rotation by $2\pi\varrho$, where $\varrho = \frac{p}{q}$, then φ is q-periodic, i.e., $\Phi^q(x) = x + p$. Consequently, the sequence $\eta_n(z)$ is q-periodic.

Notice that, by Remark 1.4, the displacement sequence even for the irrational rotation consists of precisely one element (thus it is very "regular"), although its phase sequence, i.e., the sequence $\Phi^n(x)$ mod 1, is dense in the whole interval $[0, 1]$.

2. Semi-Periodic Circle Homeomorphism

2.1. General properties

We recall definitions of a semi-periodic circle homeomorphism (after [6]) and a semi-periodic sequence (after [2]):

Definition 2.1. A circle homeomorphism with rational rotation number, which is not conjugated to a rotation is called *semi-periodic*.

Definition 2.2. A sequence $\{x_n\}$ is *semi-periodic*, if

$$\forall_{\varepsilon>0}\ \exists_{r\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ |x_{n+rk} - x_n| < \varepsilon. \tag{4}$$

Since we want to investigate asymptotic behaviour of orbits, we introduce additionally the concept of asymptotic semi-periodicity:

Definition 2.3. A sequence $\{x_n\}$ is *asymptotically semi-periodic*, if

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \exists_{r\in\mathbb{N}}\ \forall_{n>N}\ \forall_{k\in\mathbb{N}}\quad |x_{n+rk} - x_n| < \varepsilon. \tag{5}$$

There is also a simpler notion of asymptotic periodicity:

Definition 2.4. We say that a sequence $\{x_n\}$ is *asymptotically periodic*, if there exists a periodic sequence $\{a_n\}$ such that $\lim_{n\to\infty}|x_n - a_n| \to 0$.

Note that the definition of asymptotic semi-periodicity is more general since for an asymptotically semi-periodic sequence this "semi-period" r might depend on ε, whereas it does not for asymptotically periodic one.

In this section, we will see that the displacement sequence of a semi-periodic circle homeomorphism is asymptotically periodic, which is a natural consequence of the fact that each non-periodic orbit is attracted to some periodic orbit. Moreover, as we show in Theorem 2.10, for a semi-periodic homeomorphism φ with $\varrho(\varphi) = \frac{p}{q}$ and given $\varepsilon > 0$, there exists a natural number N such that every point $z \in S^1$ starting from Nq-iteration forward or from Nq-iteration backward is placed within ε-neighbourhood of a periodic orbit. An analogous property obviously holds for displacement sequences of points under φ, which is formulated in Proposition 2.11.

Proposition 2.5. *For a semi-periodic circle homeomorphism* φ, *the sequence* $\eta_n(z)$ *is asymptotically periodic* (*and thus in particular asymptotically semi-periodic*) *for any* $z \in S^1$. *Precisely*, *if* $\varrho(\varphi) = p / q$, *then for every* $z \in S^1$:

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n>N}\ \forall_{k\in\mathbb{N}}\quad |\eta_{n+kq}(z) - \eta_n(z)| < \varepsilon. \tag{6}$$

Proof. For all periodic points the statement reduces to Remark 1.5. Given a non-periodic point $z = \exp(2\pi ix) \in S^1$ there exists a periodic point $z_0 = \exp(2\pi ix_0) \in S^1$ and some $\widetilde{N}$ such that for all $n \geq \widetilde{N}$ and $i = 0, 1, \ldots, q-1$ we have $|\Phi^{nq+i}(x) - \Phi^{nq+i}(x_0)| < \varepsilon / 4$, i.e., the non-periodic orbit of z is asymptotic to the periodic orbit of z_0. Then the property (6) of the displacement sequence $\eta_n(z)$ holds for $N := \widetilde{N}q$. If p and q are relatively prime, then the "semi-period" $r = q$ is minimal. $\square$

Remark 2.6. We quoted the definitions of semi-periodic and asymptotically semi-periodic sequence in order to point out the differences between the notions of semi-periodicity in dynamical systems and in the numerical theory of sequences: Namely, the fact that the orbits $\{\Phi^n(x) \bmod 1\}_{n=1}^{\infty}$ of semi-periodic circle homeomorphisms are in general not semi-periodic as sequences (unless the orbits are purely periodic) and consequently the displacement sequence is not semi-periodic as well.

2.2. A uniform choice of $N(z)$

Given a homeomorphism φ with $\varrho(\varphi) = \frac{p}{q}$ and $\varepsilon > 0$, it does not exist N such that for $n > N$ and $k \in \mathbb{N}$ we have $|\eta_{n+kq}(z) - \eta_n| < \varepsilon$ for all $z \in S^1$. Nevertheless, it is possible to find one N that would fit all the points if we allow that for some points we consider positive iterates and for some negative.

Suppose that $\varrho(\varphi) = \frac{p}{q}$ (we admit also $q = 1$, where periodic points of φ are precisely fixed points). If z_*^- and z_*^+ are consecutive periodic points, then every point $z \in (z_*^-, z_*^+)$ is forward asymptotic under φ^q to z_*^+ and backward, i.e., under φ^{-q}, asymptotic to z_*^- or the other way around.

Suppose now that the set of periodic points $Per(\varphi)$ is finite and ordered as $Per(\varphi) = \{z^1, z^2, \dots, z^r\}, r \in \mathbb{N}$. Let us fix $\varepsilon > 0$ and $k \in \{1, 2, \dots, r\}$. Assume without the loss of generality that the two consecutive periodic points z^k and z^{k+1} (which belong to different periodic orbits if there is more than one periodic orbit) are, respectively, backward and forward attracting under φ^q for $z \in (z^k, z^{k+1})$. To clarify the notation we denote z^k by z_0^-, and z^{k+1} by z_0^+. It follows that all the points within the interval (z_i^-, z_i^+), where $z_i^- = \varphi^i(z_0^-)$, $z_i^+ = \varphi^i(z_0^+)$ for $i = 0, 1, \dots, q-1$ and $z_q^- = z_0^-$, $z_q^+ = z_0^+$, go forward under φ^q to z_i^+ and backward to z_i^-.

For a given $m \in \mathbb{N}$, we define the functions $\tau_m^+, \tau_m^- : [z_0^-, z_0^+] \to [0, \max_{0 \le i \le q-1} |z_i^+ - z_i^-|]$:

$$\tau_m^+(z) := \max_{0 \le i \le q-1} |\varphi^{mq+i}(z) - z_i^+|, \quad \tau_m^-(z) := \max_{0 \le i \le q-1} |\varphi^{-mq-i}(z) - z_{q-i}^-|,$$

with the following properties:

(i) $\tau_m^+(z)$ is strictly decreasing, $\tau_m^+(z_0^-) = \max_{0 \le i \le q-1} |z_i^+ - z_i^-|$, $\tau_m^+(z_0^+) = 0$.

(ii) $\tau_m^-(z)$ is strictly increasing, $\tau_m^-(z_0^-) = 0$, $\tau_m^-(z_0^+) = \max_{0 \le i \le q-1} |z_i^+ - z_i^-|$.

(iii) If $m' > m$, then $\tau_{m'}^+(z) < \tau_m^+(z)$ and $\tau_{m'}^-(z) < \tau_m^+(z)$ for every $z \in (z_0^-, z_0^+)$.

For $\varepsilon > 0$ and $m \in \mathbb{N}$ denote the subsets of $[z_0^-, z_0^+]$:

$$U_m^+(\varepsilon) := \{z : \tau_m^+(z) < \varepsilon\}, \quad U_m^-(\varepsilon) := \{z : \tau_m^-(z) < \varepsilon\}.$$

Then $z < z'$ and $z \in U_m^+(\varepsilon)$ implies $z' \in U_m^+(\varepsilon)$ and, analogously, if $z' < z$ and $z \in U_m^-(\varepsilon)$, then $z' \in U_m^+(\varepsilon)$. Put $a_m(\varepsilon) := \inf\{z \in (z_0^-, z_0^+) : z \in U_m^+(\varepsilon)\}$ and $b_m(\varepsilon) := \sup\{z \in (z_0^-, z_0^+) : z \in U_m^-(\varepsilon)\}$. It is clear that $U_m^+(\varepsilon) = (a_m(\varepsilon), z_0^+]$, $U_m^-(\varepsilon) = [z_0^-, b_m(\varepsilon))$, $\bigcup_{m=1}^{\infty} \overline{U}_m^+(\varepsilon) = [z_0^-, z_0^+]$ and $\bigcup_{m=1}^{\infty} \overline{U}_{m(\varepsilon)} = [z_0^-, z_0^+]$.

For fixed $\varepsilon > 0$, there exists m such that $U_m(\varepsilon) := U_m^-(\varepsilon) \cap U_m^+(\varepsilon) \neq \emptyset$. Let $\tilde{m} = \tilde{m}(\varepsilon) := \min\{m \in \mathbb{N} : U_m^-(\varepsilon) \cap U_m^+(\varepsilon) \neq \emptyset\}$. Then

$$\overline{U}^+_{\tilde{m}(\varepsilon)} \cap \overline{U}^-_{\tilde{m}(\varepsilon)} = [a_{\tilde{m}(\varepsilon)}, b_{\tilde{m}(\varepsilon)}],$$

is a closed interval with nonempty interior. We easily justify

(iv) For every $z \in (a_{\tilde{m}(\varepsilon)}, b_{\tilde{m}(\varepsilon)})$ and $m \geq \tilde{m}(\varepsilon)$, we have $|\varphi^{mq+i}(z) - \varphi^i(z_0^+)| < \varepsilon$ and $|\varphi^{-mq-i}(z) - \varphi^{q-i}(z_0^-)| < \varepsilon$, $i = 0, 1, \ldots, q-1$.

(v) For every $z \in [z_0^-, a_{\tilde{m}(\varepsilon)})$ and $m \geq \tilde{m}(\varepsilon)$, we have $|\varphi^{-mq-i}(z) - \varphi^{q-i}(z_0^-)| < \varepsilon$, $i = 0, 1, \ldots, q-1$.

(vi) For every $z \in (b_{\tilde{m}(\varepsilon)}, z_0^+]$ and $m \geq \tilde{m}(\varepsilon)$, we have $|\varphi^{mq+i}(z) - \varphi^i(z_0^+)| < \varepsilon$, $i = 0, 1, \ldots, q-1$.

Proposition 2.7. *Let $\varphi : S^1 \to S^1$ be a circle homeomorphism which has finitely many periodic points $\{z^1, z^2, \ldots, z^r\}$. Fix $\varepsilon > 0$ and consider the interval (z^k, z^{k+1}) between the two consecutive different periodic points z^k and z^{k+1} of φ.*

Suppose that z^{k+1} is attracting (under φ^q) and z^k is repelling within the interval (z^k, z^{k+1}). Then there exists a point $\tilde{z}_k \in (z^k, z^{k+1})$ with the following properties:

(1) *If $z \in B_k^+ := [\tilde{z}_k, z^{k+1})$ and for some $n \in \mathbb{N}$ $|\varphi^{-nq-i}(z) - \varphi^{q-i}(z^k)| < \varepsilon$ for all $i = 0, 1, \ldots, q-1$, then also $|\varphi^{nq+i}(z) - \varphi^i(z^{k+1})| < \varepsilon$, $i = 0, 1, \ldots, q-1$.*

(2) *If $z \in B_k^- := (z^k, \tilde{z}_k]$ and for some $n \in \mathbb{N}$ $|\varphi^{nq+i}(z) - \varphi^i(z^{k+1})| < \varepsilon$ for all $i = 0, 1, \ldots, q-1$, then also $|\varphi^{-nq-i}(z) - \varphi^{q-i}(z^k)| < \varepsilon$, $i = 0, 1, \ldots, q-1$.*

If the point z^k *is attracting and* z^{k+1} *is repelling, then* $B_k^+ := (z^k, \tilde{z}_k]$, $B_k^- := [\tilde{z}_k, z^{k+1})$ *and the analogues of* (1) *and* (2) *hold.*

The same occurs if there is only one periodic, i.e., fixed, point z_0 *but then*

$$S^1 \setminus \{z_0\} = B_0^+ \cup B_0^- \text{ and } B_0^+ \cap B_0^- = \{\tilde{z}_0\}.$$

The above proposition says that for every point $z \in B_k^+$ its positive semi-orbit $\{\varphi^n(z)\}_{n\in\mathbb{N}}$ in shorter time (in terms of number of iterates) is placed in the ε-neighbourhood of the periodic orbit $\{z^{k+1}, \varphi(z^{k+1}), \ldots, \varphi^{q-1}(z^{k+1})\}$ (i.e., for sufficiently large $n\, |\varphi^{nq+i}(z) - \varphi^i(z)| < \varepsilon$ for every $i = 0, 1, \ldots, q-1$) than its negative semi-orbit $\{\varphi^{-n}(z)\}_{n\in\mathbb{N}}$ is placed in the ε-neighbourhood of the repelling orbit $\{z^k, \varphi^{-1}(z^k), \ldots, \varphi^{-(q-1)}(z^k)\}$. Similarly, the orbits of points of B_k^- are faster, but in negative time, attracted to the ε-neighbourhood of the orbit of z^k than to the ε-neighbourhood of the orbit of z^{k+1}.

Definition 2.8. We call a one-point set $\{\tilde{z}_k\}$ the ε-basins′ edge, since it divides the whole basin B_k into the positive and negative sub-basins B_k^+ and B_k^-, respectively, and is a common border of them.

Proof of Proposition 2.7. Let us consider the interval $[a_{\tilde{m}}, b_{\tilde{m}}] \subset (z^k, z^{k+1})$. By definition $\tau_{\tilde{m}}^+ : [a_{\tilde{m}}, b_{\tilde{m}}] \to [0, \varepsilon]$ with $\tau_{\tilde{m}}^+(a_{\tilde{m}}) = \varepsilon$. Correspondingly, $\tau_{\tilde{m}}^- : [a_{\tilde{m}}, b_{\tilde{m}}] \to [0, \varepsilon]$ with $\tau_{\tilde{m}}^-(b_{\tilde{m}}) = \varepsilon$. Moreover, $\tau_{\tilde{m}}^-(a_{\tilde{m}}) < \tau_{\tilde{m}}^-(b_{\tilde{m}}) = \tau_{\tilde{m}}^+(a_{\tilde{m}})$ and $\tau_{\tilde{m}}^+(b_{\tilde{m}}) < \tau_{\tilde{m}}^+(a_{\tilde{m}}) = \tau_{\tilde{m}}^-(b_{\tilde{m}})$. There exists a unique point $\tilde{z}_k \in (a_{\tilde{m}}, b_{\tilde{m}})$ such that $\tau_{\tilde{m}}^+(\tilde{z}_k) = \tau_{\tilde{m}}^-(\tilde{z}_k)$. Now the statement of Proposition 2.7 follows from the properties (iv) – (vi). $\square$

Remark 2.9. Note that for a given ε-accuracy the basins' edge $\widetilde{z}_k \in (z_k, z_{k+1})$ is defined by $\tau^+_{\widetilde{m}_k}$ and $\tau^-_{\widetilde{m}_k}$ and thus it depends on $\widetilde{m}_k$. Since $\widetilde{m}_k$ depends on ε, the basins' edge $\widetilde{z}_k$ changes if we change the ε-accuracy of approximation.

The interesting thing is the location of ε-basins′ edge in the given interval (z^k, z^{k+1}). We carried out the numerical simulations for the two functions, $f(x) = x^2$ and $f(x) = \frac{2}{\pi}$ arcsin x, $x \in [0, 1]$, and $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.01$, and $\varepsilon_4 = 0.001$. These functions properly extended onto $\mathbb{R}$, i.e., such that in every interval $[l, l+1]$ we have a copy of $f(x)$ on $[0, 1]$ shifted l-units upward, induce orientation preserving circle homeomorphisms with a fixed point. In both cases, the fixed point $x^+ = 0$ was attracting and the fixed point $x^- = 1$ was repelling for $x \in (0, 1)$. The results of this numerical experiment are presented in Figure 1, together with the graphs of τ^+_m and τ^-_m, for $m = m(\varepsilon_1), m(\varepsilon_2), m(\varepsilon_3), m(\varepsilon_4)$. It seems that ε-basins′ edge $\widetilde{x}(\varepsilon)$ tends to x^- as $\varepsilon \to 0$ for $f(x) = x^2$ and $\widetilde{x}(\varepsilon) \to x^+$ for $f(x) = \frac{2}{\pi}$ arcsin x. Such a behaviour of the ε-basins′ edge in these examples is not surprising, when we realize that $x^+ = 0$ is a super attracting fixed point for $f(x) = x^2$ (i.e., $f'(0) = 0$) and, similarly, $x^- = 1$ is super repelling for $f(x) = \frac{2}{\pi}$ arcsin x (i.e., $f'(x) \to \infty$ with $x \to 1$). Thus for the first case attracting to 0 is "stronger" than repelling from 1 and consequently ε-basins′ edge tends 1. For the second example the opposite holds. It seems that in general the location of ε-basins′ edge with $\varepsilon \to 0$ depends on the behaviour of derivatives $f'(x^-)$ and $f'(x^+)$. Here we do not provide a rigorous proof but just the theoretical predicates: Suppose that $f \in C^1([0, 1])$. Then the value of $|f'(x^+)|$ gives the "speed" of attraction to x^+ since $f^m(x_0) \approx x^+ + f' \ (x^+)^m(x_0 - x^+)$ for large m (when $f^{m-1}(x_0)$ is sufficiently close to x^+) as follows from the Taylor expansion and the continuity of the derivative. Similarly, as $(f^{-1})'(x^-) = 1 / f'(x^-)$, $f^{-m}(x_0) \approx x^- + 1 / (f'(x^-))^m(x_0 - x^-)$ and the rate of repelling from x^- depends on $f'(x^-)$.

Suppose now without the loss of generality that $x^- = 0$ and $x^+ = 1$ for a given C^1-function $f : [0, 1] \to [0, 1]$. Using the above and the formulas for $\tau_m^+(x_0)$ and $\tau_m^-(x_0)$, we have that

$\tau_m^+ = |f'(1)|^m(1 - x_0)$ and $\tau_m^- = |\frac{1}{f'(0)}|^m x_0$. If x_0 was to be ε-basins′ edge, then $\tau_m^+(x_0) = \tau_m^-(x_0)$ yields that

$$x_0(1 + |\frac{1}{f'(0)f'(1)}|^m) = 1.$$

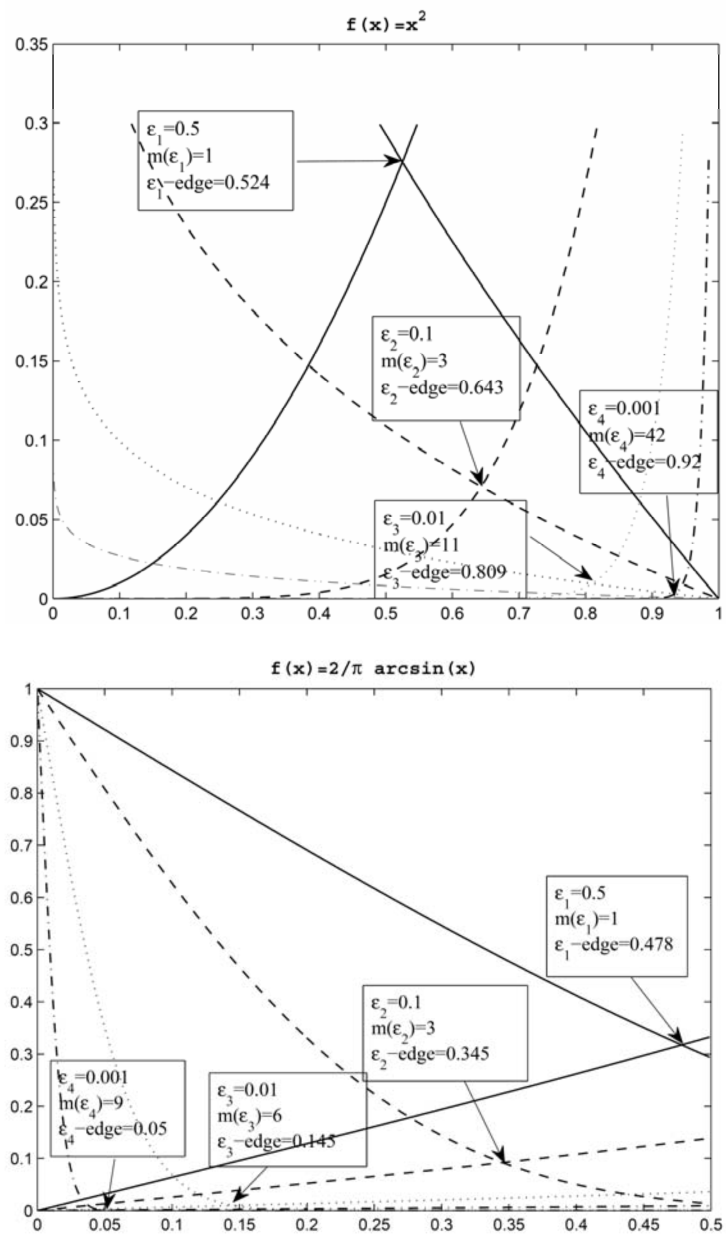

Figure 1. The $\tau_m(\varepsilon)$-function and ε-basins′ edge for $f(x) = x^2$ (the y-axis cut to [0, 0.35] for better clarity of the picture) and $f(x) = (2/\pi)$ arcsin x (the x-axis cut to [0, 0.5]).

As $\varepsilon \to 0$, $m = m(\varepsilon) \to \infty$ and the location of the basins' edge x_0 depends on the behaviour of $w^m := |f'(0)f'(1)|^m$ with $m \to \infty$. We conclude that:

• When $w^m \to \infty$, then ε-basins′ edge goes to the attracting point (since repelling from x^- is stronger that the attraction to x^+).

• When $w^m \to 0$, then ε-basins′ edge goes to the repelling point (since here the attraction is stronger).

This is consisted with the examples above and with another example $f(x) = \frac{4}{\pi}$ arctan x, $x \in [0, 1]$, which we have also investigated numerically and obtained that $\varepsilon_1 = edge = 0.497$, $\varepsilon_2 - edge = 0.409$, $\varepsilon_3 - edge = 0.269$ and $\varepsilon_1 - edge = 0.155$ with $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.01$, $\varepsilon_4 = 0.001$. Notice that here $w < 1$ and ε-basins′ edge goes to the repelling point.

However, the ε-basins′ edge for $\varepsilon \to 0$ could be as well any interior point of (z^k, z^{k+1}). probably when $w = 1$ (i.e., then $f'(x^k) = 1/f'(x^{k+1})$ up to an absolute value) or when one fixed point is super repelling and the other one is strongly attracting. As for the latter, we computed ε-basins′ edge for $f(x) = 1 - \sqrt{(1 - x^2)}$, where 0 is super attracting and 1 is super repelling, to find that ε-edge equals 0.5 for all the investigated values of ε (but in general in such a case the asymptotic basins' edge does not to be exactly a geometrical middle of the interval; here it is so be cause the function is very "symmetrical").

Going back to the displacement sequence, we prove the following:

Theorem 2.10. *Let* $\varphi : S^1 \to S^1$ *be a homeomorphism with a rotation number* $\varrho(\varphi) = \frac{p}{q}$.

For $\varepsilon > 0$, *there exists* $N = N(\varepsilon)$ *such that for every point* $z \in S^1$ *at least one of the following two conditions is satisfied*:

(1) *there exists a periodic point* $z_0^+ \in Per(\varphi)$ *such that* $|\varphi^{nq+i}(z) - \varphi^i(z_0^+)| < \varepsilon$ *for all* $n \geq N$ *and* $i = 0, 1, \dots, q-1$;

or

(2) *there exists a periodic point* $z_0^- \in Per(\varphi)$ *such that* $|\varphi^{-nq-i}(z) - \varphi^{-i}(z_0^-)| < \varepsilon$ *for all* $n \geq N$ *and* $i = 0, 1, \dots, q-1$,

i.e., after Nq *iterations forward or* Nq *iterations backward we are always* ε*-close to one of the periodic orbits.*

Proof. Assume firstly that φ has r different periodic points $z^1, z^2, \dots, z^r$. For each $1 \leq k \leq r$, we apply the properties (iv) – (vi) with $z_0^- = z^k$ and $z_0^+ = z^{k+1}$ (or $z_0^- = z^{k+1}$ and $z_0^+ = z^k$ if z^k is attracting and z^{k+1} repelling for $z \in (z^k, z^{k+1})$. Set $N = \widetilde{m} = \widetilde{m}_k$. Then $z \in B_k^+$ satisfies (1) and $z \in B_k^-$ satisfies (2). Every point $z \in (a_{\widetilde{m}}(\varepsilon), b_{\widetilde{m}}(\varepsilon))$ fulfills both (1) and (2). Now, since $S^1 = [z^1, z^2] \cup \dots [z^{k+1}, z^k] \cup [z^r, z^1]$, it is enough to take $N = \max_{1 \leq k \leq r+1} \widetilde{m}_k$ (where $\widetilde{m}_{r+1}$ corresponds to the interval $[z^r, z^1]$) to get the statement.

Suppose now that $\# Per(\varphi) = \infty$. $Per(\varphi)$ is closed thus compact subset of S^1. Fix $\varepsilon > 0$. The proof will be carried out in the following steps:

(1) Let z_0 be a periodic point with the orbit $\mathcal{O} = \{z_0, z_1, \dots, z_{q-1}\}$ for which there exists another periodic point z_0' with the orbit $\mathcal{O} = \{z_0', z_1', \dots, z_{q-1}'\}$ such that for every $i = 0, 1, \dots, q-1$, z_i and z_i' are consecutive periodic points, $z_i' > z_i$ and for at least one $i_* \in \{0, 1, \dots, q-1\}$ we have $|z_i' - z_i| \geq \varepsilon$. If it is not possible to find such a point z_0, then the distance between any two consecutive periodic points is smaller than ε and the hypothesis of Theorem 2.10 is satisfied in a trivial way.

(2) Notice that the number of intervals (z, z') between consecutive periodic points z and z' such that $|z' - z| \geq \varepsilon$ is finite. Consequently, the number of pairs $\{\mathcal{O}, \mathcal{O}'\}$ of periodic orbits $\mathcal{O}$ and $\mathcal{O}'$ such as in (1) is finite. Denote as D^{ε} the collection of all such ordered pairs $\{\mathcal{O}, \mathcal{O}'\}$.

(3) Let now z be an arbitrary point on S^1. If $z \in Per(\varphi)$, there is nothing to prove. If $z \notin Per(\varphi)$ and z does not lie in any interval $(z_* \ z'_*)$, where $z_* \in \mathcal{O}$ and $z'_* \in \mathcal{O}'$ with the pair of periodic orbits $\{\mathcal{O}_*, \mathcal{O}'_*\} \in D^{\varepsilon}$, then every point of the full orbit $\{\varphi^i(z)\}_{i \in \mathbb{Z}}$ belongs to some interval between the two periodic points with length smaller than ε. As a result, conditions (1) and (2) of Theorem 2.10 are satisfied in a trivial way with arbitrary $N \in \mathbb{N} \cup \{0\}$. If $z \notin Per(\varphi)$ but there exist periodic points z_* and z'_* such that $z \in (z_*, z'_*)$ and $z_* \in \mathcal{O}_*$, $z'_* \in \mathcal{O}'_*$ with a pair of orbits $\{\mathcal{O}_*, \mathcal{O}'_*\} \in D^{\varepsilon}$, then at least one of the conditions (1) or (2) holds for z with $N = N_{\max}$, where $N_{\max}$ is the "universal" N derived, as in the first part of the proof, for the finite collection of all the intervals $\{(z^i, z'^i)\}$ between the two consecutive periodic points whose orbits $\mathcal{O}_i$ and $\mathcal{O}'_i$ form pairs $\{\mathcal{O}_i, \mathcal{O}'_i\} \in D^{\varepsilon}$.

Consequently, it is enough to take $N = N_{\max}$ for an arbitrary $z \in S^1$. □

Note that in general a number N satisfying the statement of Theorem 2.10 could be found by considering, instead of ε-basins$'$ edge, just the geometrical middles $\hat{z}_k$ of the intervals $\hat{I}_k$ between periodic points (with union $\bigcup_k \hat{I}_k$ giving the whole of S^1), computing the corresponding numbers $\hat{N}_k$ such that the iterates $\varphi^{-nq-i}(\hat{z}_k)$ and $\varphi^{nq+i}(\hat{z}_k)$ are placed in the ε-neighbourhood of periodic orbits for $n > \hat{N}_k$ and $i = 0, 1, \dots, q - 1,$

and then taking $N = \max_k \hat{N}_k$. However, given $\varepsilon > 0$ the notion of the ε-basins′ edge says how to find the smallest $N \in \mathbb{N}$ with these properties.

Proposition 2.11. *Let* $\varrho(\varphi) = \frac{p}{q}$. *Then for every* $\varepsilon > 0$ *there exists* N *such that for every* $z \in S^1$ *the sequence* $\{\eta_n(z)\}_{n=-\infty}^{\infty}$ *satisfies at least one of the following statements*:

$$\forall_{n>N}\ \forall_{l\in\mathbb{N}}\ |\eta_{n+lq}(z) - \eta_n(z)| < \varepsilon, \tag{1}$$

or

$$\forall_{n>N}\ \forall_{l\in\mathbb{N}}\ |\eta_{-(n+lq)}(z) - \eta_n(z)| < \varepsilon. \tag{2}$$

Proof. The proposition is a direct consequence of Theorem 2.10. □

3. Homeomorphisms with Irrational Rotation Number

Let now $\varrho(\varphi)$ be irrational. If φ is not transitive by $\Delta \subset S^1$ denote the unique minimal set of φ (a Cantor type set).

We have to introduce the following definition:

Definition 3.1. We say that a sequence $\{a_n\}$, $n \in \mathbb{N}$, is almost strongly recurrent if it satisfies

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ \exists_{i\in\{0,1,\dots,N\}}\ |a_{n+k+i} - a_n| < \varepsilon.$$

For an almost strongly recurrent sequence we require that for each n the set of returns $\mathcal{R} := \{r : r = k + i\}$ of a_n to its ε-neighbourhood may depend on n, although for all n its gaps are bounded by the same N. Note that in literature there is also a notion of an almost periodic sequence ([2]), for which with given $k \in \mathbb{N}$, the index $i \in \{0, 1, \dots, N\}$ can be chosen uniformly for all $n \in \mathbb{N}$ and thus almost periodicity of a sequence is a stronger property.

Let then (X, φ) be a discrete dynamical system, where (X, d) is a metric space and $\varphi \in C^0(X)$.

Definition 3.2 ([9]). A point $x \in X$ is almost periodic if

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ \exists_{i\in\{0,1,\dots,N\}}\ d(\varphi^{k+i}(x), x) < \varepsilon,$$

i.e., when for any neighbourhood U of x the set of numbers n such that $\varphi^n(x) \in U$ is relatively dense in $\mathbb{N}$.

Sometimes an almost periodic point is called also *strongly recurrent*. Recall the following theorem of Gottschalk:

Theorem G ([9]). *Let X be a compact metric space.*

Then the closure of the orbit of any almost periodic point is a minimal set. Conversely, all points of any minimal set are almost periodic.

Proposition 3.3. *Let* $\varphi : S^1 \to S^1$ *be a homeomorphisms with an irrational rotation number.*

(1) *If* φ *is transitive, then for all* $z \in S^1$ *the displacement sequenc e* $\{\eta_n(z)\}$ *is almost strongly recurrent, i.e.,*

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}\cup\{0\}}\ \exists_{i\in\{0,1,\dots,N\}}\ |\eta_{n+k+i}(z) - \eta_n(z)| < \varepsilon.$$

(2) *If* φ *is not-transitive, then the sequence* $\{\eta_n(z)\}$ *is almost strongly recurrent for all* $z \in \Delta$, *where* Δ *is the minimal invariant set of* φ.

The theorem will follow easily from the lemma.

Lemma 3.4. *Let* (X, d) *be a compact metric space and* (X, φ), *where* $\varphi \in C^0(X)$, *a discrete dynamical system with X being a minimal invariant set.*

Given $x \in X$ *and* $\varepsilon > 0$, *the set of return-times of the orbit* $\{\varphi^j(x)\}_{j=1}^{\infty}$ *to the* ε*-neighbourhood of any point* $\varphi^n(x)$ *of the orbit of x is relatively dense with gaps bounded uniformly for all n*:

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{k\in\mathbb{N}\cup\{0\}}\ \forall_{k\in\mathbb{N}\cup\{0\}}\ \exists_{i\in\{1,2,\dots,N\}}\ d(\varphi^{n+k+i}(x),\ \varphi^{n}(x)) < \varepsilon.$$

Proof. By compactness of X, there exists m such that the open balls $B_l = B(\varphi^l(x), \varepsilon/2)$ for $l = 0, 1, \dots, m$ cover X. There exist numbers $N_0, N_1, N_2, \dots, N_m$ such that for any $k \in \mathbb{N}\cup\{0\}$, if $\varphi^k(\varphi^l(x)) \in B_l$, then also $\varphi^{k+i_l}(\varphi^l(x)) \in B_l$ for some $i_l \in \{1, 2, \dots, N_l\}$. In this case, $d(\varphi^{l+k+i_l}(x), \varphi^l(x)) < \varepsilon$. Setting $N = \max_{l\in\{0,1,\dots,m\}} N_l$, we obtain that

$$\forall_{l\in\{0,1,\dots,m\}}\ \forall_{k\in\mathbb{N}\cup\{0\}}\ \exists_{i\in\{1,2,\dots,N\}}\ d(\varphi^{l+k+i}(x),\ \varphi^{l}(x)) < \varepsilon.$$

Consider now an arbitrary point $\varphi^n(x)$, $n > m$, of the orbit $\{\varphi^j(x)\}_{j=1}^{\infty}$. There exists $l \in \{0, 1, \dots, m\}$ such that $\varphi^n(x) \in B_l$ and $\varphi^n(x) = \varphi^j$ $(\varphi^l(x))$ for some $j \in \mathbb{N}$, which is then a returning time of a point $\varphi^l(x)$ to B_l. Consequently, there exists $\widetilde{k}_1 \in \{1, 2, \dots, N\}$ such that $\varphi^{j+k_1}(\varphi^l(x)) = \varphi^{\widetilde{k}_1}(\varphi^n(x)) \in B_l$. Similarly, there exists $\widetilde{k}_2 \in \{1, 2, \dots, N\}$ such that $\varphi^{j+\widetilde{k}_1+\widetilde{k}_2}(\varphi^l(x)) = \varphi^{\widetilde{k}_1+\widetilde{k}_2}(\varphi^n(x)) \in B_l$, etc. The sequence of numbers $k_r := \widetilde{k}_1 + \widetilde{k}_2 + \dots + \widetilde{k}_r, r \in \mathbb{N}$, is relatively dense in $\mathbb{N}$ and for every $r\, d(\varphi^{k_r}(\varphi^n(x)), \varphi^n(x)) < \varepsilon$. □

Proof of Proposition 3.3. Let $z \in \Delta$ (where $\Delta = S^1$ if φ is transitive) and $\varepsilon > 0$ be arbitrary. There exists $\delta < \varepsilon/2$ such that for every $z_1, z_2 \in S^1$ we have $|\varphi(z_1) - \varphi(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$. On the account of Lemma 3.4, there exists N such that for all integers $n > 0$ and $k \geq 0$ and some $i \in \{1, 2, \dots, N\}$ we have $|\varphi^{(n-1)+k+i}(z) - \varphi^{n-1}(z)| < \delta$. Then

$$|\varphi^{n+k+i}(z) - \varphi^{n+k+i-1}(z) - \varphi^{n}(z) + \varphi^{n-1}(z)| < \varepsilon/2 + \delta < \varepsilon,$$

which implies, with a little effort, that $|\eta_{n+k+i}(z) - \eta_n(z)| < \varepsilon$. □

4. Discussion

In the end let us make a few comments.

Remark 4.1. Note that Proposition 3.3 applies in more general setting where (X, φ) is a minimal dynamical system on a compact metric space (X, d) and the displacement sequence is defined simply as $\eta_n(x) := d(\varphi^n(x)), \varphi^{n-1}(x))$.

Remark 4.2. Obviously, the corresponding results for the "phase"-sequence, i.e., the sequence $\Phi^n(x) \bmod 1 \in S^1$, are also true, and, as it is visible in the proofs, in fact the presented properties of the displacement sequence follow from the properties of the sequence of iterates of $\varphi = \Phi$ mod 1. Nevertheless, not all of the properties of the displacement sequence are such natural. For instantce, in [13], we investigated the distribution μ_Ψ of the displacement sequence with respect to the unique ergodic probability measure μ (in case of irrational rotation number) and for the existence of the density (with respect to the Lebesque measure) of μ_Ψ it is not sufficient that φ is conjugated to the rotation via a C^1-diffeomorphism (which actually gives that the invariant measure μ has density): we have to assume additionally that the set of critical points of the displacement function $\Psi := \Phi - Id$ is of Lebesque measure zero. In [13], we also provided the effective formula for the density of μ_Ψ and proved the rigorous result concerning the approximation of μ_Ψ by sample displacement distributions along the orbits of another circle homeomorphism, not necessary with irrational rotation number, close enough (in C^0-topology) to φ.

In the forthcoming paper "On the interspike-intervals of periodically-driven integrate-and-fire models", we present the implications of the results on the displacement sequence obtained here and in [13] to integrate-and-fire models. In particular, we show that the sequence of interspike-intervals for the models of leaky integrate-and-fire type, i.e., models of the form $\dot{x} = -\sigma x + f(t)$ (together with the resetting mechanism), exhibits exactly the same properties as the displacement sequence of an orientation preserving circle homeomorphism φ whenever the input function f is locally integrable, periodic and satisfies $f(t) - \sigma > \zeta$ a.e. for some $\zeta > 0$ (in this case φ is the projection of the induced firing map Φ onto S^1).

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