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ON THE LAFORGIA-NATALINI'S INEQUALITY FOR THE RIEMANN ZETA FUNCTION

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Abstract

We obtain the inequality $\zeta(s)\zeta(s+2) > [\zeta(s+1)]^2$, $s > 1$, for the Riemann zeta function, which implies the inequality of Laforgia-Natalini [1].

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1. Introduction

Laforgia-Natalini [1, 2] employ a generalization of the Schwartz inequality to deduce the following inequality for the Riemann zeta function [3]:

$$\frac{(s+1)\zeta(s)}{s\zeta(s+1)} > \frac{\zeta(s+1)}{\zeta(s+2)}, \quad s > 1. \tag{1}$$

Here, we use known properties of $\zeta(s)$ [4] to show that

$$\frac{\zeta(s)}{\zeta(s+1)} > \frac{\zeta(s+1)}{\zeta(s+2)}, \quad s > 1, \tag{2}$$

which is stronger than (1), that is, (2) implies (1). It is unknown a closed expression for the Riemann zeta function valued at positive odd integers, then we consider very useful to obtain from (2) a narrow inequality for $\zeta(2n+1)$, $n = 1, 2, \ldots,$ which implies a corresponding inequality for Faulhaber [5]-Bernoulli [6] numbers.

2. Formula of Titchmarsh

In [4] page 6, we find the expression

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Phi(n)}{n^s}, \quad s > 2, \tag{3}$$

where $\Phi(n)$ is the amount of numbers less than n and prime to n. Then for $s > 1$ are valid the relations

$$\frac{\zeta(s)}{\zeta(s+1)} = \sum_{n=1}^{\infty} \frac{\Phi(n)}{n^{s+1}} \quad \text{and} \quad \frac{\zeta(s+1)}{\zeta(s+2)} = \sum_{n=1}^{\infty} \frac{\Phi(n)}{n^{s+2}},$$

whose all terms are positive and $\frac{1}{n^{s+1}} > \frac{1}{n^{s+2}}$, thus each term in $\frac{\zeta(s)}{\zeta(s+1)}$ is greater than the corresponding term in $\frac{\zeta(s+1)}{\zeta(s+2)}$, therefore (2) is correct for $s > 1$. Besides, $\frac{s+1}{s} > 1$, then (2) implies (1).

3. Inequalities for $\zeta(2n+1)$, $n = 1, 2, \ldots$

If in (2), we use $s = 2n$ and the result of Euler (1735) [3, 4]

$$\zeta(2n) = -(-1)^n \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad n = 1, 2, \ldots, \tag{4}$$

with the Faulhaber [5]-Bernoulli [6] numbers

$$B_0 = 1, \ B_2 = \frac{1}{6}, \ B_4 = B_8 = \frac{-1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{10} = \frac{5}{66}, \ldots, \tag{5}$$

we deduce the following inequality for Riemann zeta function at odd integers:

$$\zeta(2n+1) < \frac{(2\pi)^{2n+1}}{2^{\frac{3}{2}}(2n)!} \left[-\frac{B_{2n} B_{2n+2}}{(n+1)(2n+1)} \right]^{1/2}, \quad n = 1, 2, \ldots. \tag{6}$$

For example, if in (6), we employ $n = 1, 2$ and the values (5), then

$$\zeta(3) < 1.334\,297\,702, \qquad \zeta(5) < 1.049\,330\,278,$$

in accordance with the values $\zeta(3) = 1.202\,056\,903$ and $\zeta(5) = 1.036\,927755$ reported in the literature.

In [4] page 191 is the inequality

$$\frac{1}{\zeta(s)} \le \frac{\zeta(s)}{\zeta(2s)}, \quad s > 1, \tag{7}$$

where we may use $s = 2n + 1$ and (4) to obtain that

$$(2\pi)^{2n+1} \left[\frac{B_{4n+2}}{2(4n+2)!} \right]^{1/2} \le \zeta(2n+1), \quad n = 1, 2, 3, \ldots, \tag{8}$$

which for $n = 1, 2$ implies the correct inequalities

$$1.008\,634\,256 \le \zeta(3), \qquad 1.000\,497\,164 \le \zeta(5).$$

Thus, the expressions (6) and (8) give us an interval for $\zeta(2n+1)$ and also the following inequality for Faulhaber-Bernoulli numbers:

$$2(n+1)(2n)!\ B_{4n+2} < -4^n(4n+1)!!\,B_{2n}B_{2n+2}. \tag{9}$$

For example, (9) can be verified with the values (5). The relations (2), (6), (8), and (9) are not in the literature.

4. Conclusion

Employing known relations [4] for Riemann zeta function it is possible to obtain, in elementary manner, the inequality (2), which implies the result of Laforgia-Natalini [1]. Besides, the approach here presented leads to an inequality for $\zeta(2n+1)$, $n = 1, 2, \ldots,$ expressions (6) and (8), and the corresponding inequality (9) for Bernoulli numbers.

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