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NONOBLATENESS OF A GENERATING CONE IN *SH*-SPACE AND ITS APPLICATION

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Abstract

The concept of a nonoblate cone in a Banach space is one of the most important ideas in the theory of ordered normed linear spaces. In connection with the introduction, the new class of *SH*-spaces by Smirnov (the *H*-spaces as Souslin spaces earlier), the problem of clarifying the role of the concept of nonoblateness of a cone in such spaces arises naturally. In the present paper, we will obtain a theorem about the nonoblateness of a generating cone in an *SH*-space and demonstrate a series of its applications to questions of differentiability with respect to a cone and of the continuity of a positive operator. This will allow us to obtain a theorem on the existence of a saddle point of the Lagrange function for linear optimization problems in *SH*-spaces.

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1. Nonoblate Cones in *SH*-Spaces

We recall [2] that a cone K in a locally convex space (LCS) X is said to be *nonoblate* if for each neighbourhood of zero U, there exists a neighbourhood of zero V for which $V \subset U \cap K - U \cap K$. The theory of differentiation in an LCS as developed in [1] is used systematically. All topological vector spaces considered are assumed to be separated and locally convex.

Let (G, τ) be a locally convex metric topological vector group (TVG) and K be a closed generating cone in G. We will denote by d a quasinorm defining the topology τ, i.e., a nonnegative functional on G, which satisfies the conditions:

(a) $0 \leq d(x) \leq 1 \quad (x \in G)$;

(b) $d(\lambda x) \leq d(x) \quad (|\lambda| \leq 1,\ x \in G)$;

(c) $d(x_1 + x_2) \leq d(x_1) + d(x_2) \quad (x_1,\ x_2 \in G)$.

The quasinorm

$$\widetilde{d}(x) = \inf\{d(u) + d(v) \colon x = u - v,\ u, v \in K\},$$

defines on G the topology $\widetilde{\tau}$ of a locally convex TVG in which a base of absolutely convex neighbourhoods of zero is formed by the sets

$$V_n = K \cap U_n - K \cap U_n \quad (n = 1, 2, \ldots),$$

where $\{U_n : n = 1, 2, \ldots\}$ is a base of absolutely convex neighbourhoods of zero in the topology τ. It is clear that $\tau \leq \widetilde{\tau}$.

Proposition 1. *If* (G, τ) *is a complete TVG, then it follows from convergence in* (G, τ) *of the series*:

$$x = \sum_{n=1}^{\infty} x_n, \tag{1}$$

that

$$\tilde{d}(x) \leq \sum_{n=1}^{\infty} \tilde{d}(x_n). \tag{2}$$

Proof. Suppose that the series (1) converges in (G, τ) and the right-hand side of inequality (2) is finite. Then for every $\epsilon > 0$, there exist sequences $u_n \in K$ and $v_n \in K$ for which $x_n = u_n - v_n$ and

$$d(u_n) + d(v_n) \leq \tilde{d}(x_n) + 2^{-n}\epsilon.$$

Since (G, τ) is a complete TVG, it follows from this that there exist elements $u, v \in K$ for which $x = u - v$ and

$$\tilde{d}(x) \leq d(u) + d(v) \leq \sum_{n=1}^{\infty} [d(u_n) + d(v_n)] \leq \sum_{n=1}^{\infty} \tilde{d}(x_n) + \epsilon.$$

Inequality (2) follows from this since $\epsilon > 0$ is arbitrary. The proposition is proved.

From Proposition 1 and the completeness of (G, τ), we deduce that any series (1) for which the right-hand side of inequality (2) is finite converges in (G, τ). Hence we have

Proposition 2. *The TVG* $(G, \tilde{\tau})$ *is complete.*

Proof. Let (x_n) be a fundamental sequence in $(G, \tilde{\tau})$. We choose a subsequence (x_{n_k}) such that

$$\tilde{d}(x_{n_{k+1}} - x_{n_k}) < 2^{-k} \quad (k = 1, 2 \ldots).$$

Then

$$\sum_{k=1}^{\infty} \tilde{d}(x_{n_{k+1}} - x_{n_k}) < \infty,$$

and consequently, the series

$$x_{n_1} + \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}),$$

converges in $(G, \tilde{\tau})$. In the other words, the subsequence (x_{n_k}), and along with it also the sequence (x_n), converge in $(G, \tilde{\tau})$. The proposition is proved.

Theorem 1. *Let* (X, τ^*) *be an SH-space and K be a generating closed cone in X. Then K is a nonoblate cone.*

Proof. Let (X, τ^*) be an *SH*-space and *K* be a generating closed cone in *X*. Let

$$X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} X_{n_1 n_2 \ldots n_k},$$

be such that τ^* is the strongest locally convex topology on *X* for which all the embeddings of the locally convex metric TVGs $X_{(\nu)} (\nu \in \mathcal{P})$ in the space (X, τ^*) are continuous. Without loss of generality, it can be assumed that the spaces $X_{n_1 n_2 \ldots n_k} (n_k, k = 1, 2, \ldots)$ are seminormed and the embeddings

$$X_{n_1 n_2 \ldots n_{k+1}} \to X_{n_1 n_2 \ldots n_k} \quad (k = 1, 2, \ldots; \nu \in \mathcal{P}),$$

are continuous. Here $\mathcal{P}$ is a subset of $\mathcal{N}^*$, the set of sequences of positive integers.

Let $\nu = (n_1, n_2, \ldots) \in \mathcal{P}$. We will denote by $\{U_{n_1 n_2 \ldots n_k} : k = 1, 2, \ldots\}$ the family of absolutely convex neighbourhoods of zero in a base for the space $X_{(\nu)}$, which are such that for all $k = 1, 2, \ldots,$ the set $U_{n_1 n_2 \ldots n_k}$ is a neighbourhood of zero in $X_{n_1 n_2 \ldots n_k}$ and $2U_{n_1 n_2 \ldots n_{k+1}} \subset U_{n_1 n_2 \ldots n_k}$. Then the sets

$$V_{n_1 n_2 \ldots n_k} = U_{n_1 n_2 \ldots n_k} \cap K - U_{n_1 n_2 \ldots n_k} \cap K \quad (k = 1, 2, \ldots),$$

are absolutely convex and their linear hulls $L(V_{n_1 n_2 \ldots n_k}) = Y_{n_1 n_2 \ldots n_k}$ can be given seminorm topologies in such a way that for each $k = 1, 2, \ldots,$ the sets $\epsilon V_{n_1 n_2 \ldots n_k} (\epsilon > 0)$ form a base of neighbourhoods of zero. It is not difficult to see that

$$X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} Y_{n_1 n_2 \ldots n_k},$$

and moreover, the sequence $V_{n_1 n_2 \ldots n_k}$ forms a base of absolutely convex neighbourhoods of zero for some TVG $Y_{(\nu)}$. Since the space $X_{(\nu)}$ is complete, then by Proposition 2, the TVG $Y_{(\nu)}$ is also complete.

Now, let us consider on X the strongest locally convex topology σ^* for which all the embeddings of the spaces $Y_{(\nu)} (\nu \in \mathcal{P})$ in the space (X, σ^*) are continuous. Then (X, σ^*) is an *SH*-space and moreover $\tau^* \leq \sigma^*$. By the Closed Graph Theorem for *SH*-spaces, we have the inequality $\sigma^* \leq \tau^*$. The assertion of the theorem now follows since by construction the cone K is nonoblate in (X, σ^*). The theorem is proved.

Corollary 1. *Let K be a generating closed cone in a sequentially complete bornological SH-space (X, τ). Then K is a nonoblate cone.*

This assertion follows from Theorem 1 and Proposition 7.3.5 of [4].

2. Compact Differentiability with Respect to a Cone

Let X and Y be LCSs, K be a closed cone in X and $L(X, Y)$ be the vector space of all continuous linear mappings from X to Y. We will denote by β (resp., β_c) the system of all bounded (resp., compact) subsets of the space X, and by β_k (resp., β_c^k) the system of all bounded (resp., compact) subsets of the cone K. Let $L_\beta(X, Y)$ (resp., $L_{\beta_c}(X, Y)$) be the LCS obtained by giving the space $L(X, Y)$ the topology of uniform convergence on the sets of the system β (resp., β_c).

We will say (see also [2]) that the operator $A : X \to Y$ is *differentiable at the point* $x_0 \in X$ *in the directions of the cone K*, if the function $y(t) = A(x_0 + th)$ is differentiable with respect to t at the point $t = 0$ for all $h \in K$. If the derivative $y'(0)$ is representable in the form $y'(0) = A'(x_0)h (h \in K)$, where $A'(x_0) \in L(X, Y)$, then we will call the linear operator $A'(x_0)$ the *weak derivative with respect to the cone K at the point* x_0.

If the identity

$$\lim_{t\to 0} \frac{y(t) - y(0)}{t} = A'(x_0)h,$$

is satisfied uniformly with respect to $h \in B$ for each B from β_k (resp., β_c^k), then we will call $A'(x_0)$ the *bounded* (resp., *compact*) *derivative with respect to the cone K at the point* x_0. Mappings which have a weak, bounded or compact derivative with respect to a cone will be called weakly, boundedly or compactly differentiable with respect to the cone.

Let (X, τ) be a separated sequentially complete bornological *SH*-space, i.e.,

$$X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} X_{n_1 n_2 \ldots n_k},$$

and each space $X_{(\nu)}$ $(\nu \in \mathcal{P})$ is a locally convex complete metric TVG, which is continuously embedded in (X, τ). The topology τ of the space X induces on each space

$$X_\nu = \bigcap_{k=1}^{\infty} X_{n_1 n_2 \ldots n_k},$$

a locally convex topology $\tilde{\tau}_\nu$ which in general is different from the topology τ_ν of the Fréchet space X_ν $(\nu \in \mathcal{P})$. We will assume that $\tau_\nu = \tilde{\tau}_\nu$ for each $\nu \in \mathcal{P}$.

Theorem 2. *Suppose that for the operator* $A : X \to Y$ *the weak derivative* $A'(x)$ *with respect to a generating closed cone K is a continuous mapping into* $L_{\beta_c}(X, Y)$ *on an open neighbourhood U of the point x. Then* $A'(x)$ *is the compact derivative of the operator A at the points* $x \in U$.

Proof. By Corollary 1, the cone K is nonoblate in the space (X, τ) and we have the identity

$$X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} Y_{n_1 n_2 \ldots n_k},$$

where the $Y_{n_1 n_2 \ldots n_k}$ $(n_k,\ k = 1, 2, \ldots)$ are seminormed spaces and the cone K is nonoblate in each locally convex TVG $Y_{(\nu)} (\nu \in \mathcal{P})$. We have to show that

$$\lim_{\delta \to 0} \frac{A(x + \delta h) - A(x)}{\delta} = A'(x)h, \tag{3}$$

where $x \in U$ and convergence is uniform with respect to all $h \in B$ for every $B \in \beta_c$.

Let $x \in U$, $B \in \beta_c$ and let W be a convex neighbourhood of zero in the space Y. Since the space (X, τ) is sequentially complete, the set B is contained and bounded in some space Y_ν, where $\nu \in \mathcal{P}$. By the Closed Graph Theorem, there exists $\nu' \in \mathcal{P}$ such that $Y_\nu \subset Y_{\nu'}$. But $\tau_\nu = \tilde{\tau}_\nu$; therefore, the set B is compact in Y_ν and thus it is compact in $Y_{\nu'}$. By

Corollary 1 of [3], there is a sequence (h_n) converging to zero in $Y_{\nu'}$ such that B is contained in the closed absolutely convex hull of (h_n). Because of the nonoblateness of the cone K in the space $Y_{(\nu')}$, there exist sequences $(u_n) \subset K$ and $(v_n) \subset K$ for which $h_n = u_n - v_n$ and $u_n \to 0$, $v_n \to 0$ as $n \to \infty$ in the space $Y_{(\nu')}$. Hence it follows that $B \subset S - S$, where S is compact in (X, τ) and $S \in \beta_c^k$.

Choose $\delta_0 > 0$ such that $x + \delta h \in U$, $x + \delta u \in U$, and $x + \delta v \in U$, where $|\delta| \leq \delta_0$ and $h = u - v$, $h \in B$, $u, v \in S$. We introduce the notation

$$\omega(x, \delta h) = A(x + \delta h) - A(x) - A'(x)\delta h.$$

It is obvious that

$$\omega(x, \delta h) = A(x + \delta h) - A(x + \delta h + \delta v) + A(x + \delta h + \delta v) - A(x) - A'(x)\delta h.$$

Hence by the continuity of $A'(x)$ at the point x, we obtain the following identities:

$$\begin{aligned}
\omega(x, \delta h) &= -\int_0^1 A'(x + \delta h + t\delta v)\delta v\, dt + \int_0^1 A'(x + t(\delta h + \delta v))(\delta h + \delta v)\, dt \\
&\quad - \int_0^1 A'(x)\, \delta h dt \\
&= \int_0^1 [A'(x) - A'[x + \delta h + t\delta v]]\delta v\, dt \\
&\quad + \int_0^1 [A'(x + t\delta u) - A'(x)]\delta u\, dt. \qquad (4)
\end{aligned}$$

Again, by the continuity of $A'(x)$, there exists a neighbourhood of zero P in the space (X, τ) such that for all $u, v \in S$, we have the inclusions

$$[A'(x) - A'(x + P)]v \subset \frac{1}{2} W,$$

and

$$[A'(x+P)-A'(x)]u \subset \frac{1}{2}W.$$

Since the set S is bounded in (X, τ), there exists $\delta_W > 0$ such that for $|\delta| < \delta_W$, we have (as a result of (4)) the inclusions

$$\frac{\omega(x, \delta h)}{\delta} \subset \frac{1}{2}W + \frac{1}{2}W = W.$$

Now (3) follows from these inclusions. The theorem is proved.

Corollary 2. *Let* (X, τ) *be the strict inductive limit of the sequence* $\{X_n : n = 1, 2, \ldots\}$ *of Fréchet-Montel spaces and let K be a generating closed cone in* (X, τ). *Then if the weak derivative* $A'(x)$ *with respect to the cone K of the operator* $A : X \to Y$ *is a continuous mapping into* $L_\beta(X, Y)$ *on the open neighbourhood U of the point x, it is the bounded derivative of the operator A at the points* $x \in U$.

3. The Lagrange Function in *SH*-Spaces

In this section, we give the application already mentioned of Theorem 1 to the linear optimization problem in an LCS. Suppose that it is required to minimize the functional $f(x)$ under the condition $Ax \geq y_0$, where X and Y are LCSs, $A : X \to Y$ is a continuous linear operator, f is a continuous linear functional on X; (inequalities in Y are to be understood in the sense of the ordering defined by the cone K).

We recall [5] that a point $(x_0, y_0') \in X \times K_{Y'}$ is called a *saddle point* of the Lagrange function

$$H(x, y') = f(x) - y'(Ax - y_0),$$

if

$$H(x, y_0') \geq H(x_0, y_0') \geq H(x_0, y') \quad (x \in X,\ y' \in K_{Y'}).$$

Below we denote by Y_0 the linear hull in Y of an element y_0 and the subspace AX and by M the set $\{x : Ax \geq y_0\}$.

Theorem 3. *Let Y be a sequentially complete bornological SH-space and let K_Y be a generating closed cone in Y; suppose moreover that $Y = Y_0 - K_Y$. Then, the functional f attains a minimum on the set M if and only if the corresponding Lagrange function $H(x, y')$ has a saddle point (x_0, y'_0) $(x \in X, y' \in K_{Y'})$.*

For the proof, it is enough to refer to Corollary 1 and Theorem 9 of [5].

4. Conclusion

Using the closed graph theorem is the important resource for applying of space Y. Such condition for space Y is being *SH*-space of Smirnov. In particular, this class contains of Fréchet spaces and spaces $D'(\mathbb{R}^n)$ of generalized functions. So such approach lead to an expansion of mathematical models for economic tasks of optimum control in locally convex spaces.

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