NONOBLATENESS OF A GENERATING CONE IN
SH-SPACE AND ITS APPLICATION

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Abstract

The concept of a nonoblate cone in a Banach space is one of the most important ideas in the theory of ordered normed linear spaces. In connection with the introduction, the new class of SH-spaces by Smirnov (the H-spaces as Souslin spaces earlier), the problem of clarifying the role of the concept of nonoblateness of a cone in such spaces arises naturally. In the present paper, we will obtain a theorem about the nonoblateness of a generating cone in an SH-space and demonstrate a series of its applications to questions of differentiability with respect to a cone and of the continuity of a positive operator. This will allow us to obtain a theorem on the existence of a saddle point of the Lagrange function for linear optimization problems in SH-spaces.

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1. Nonoblate Cones in $SH$-Spaces

We recall [2] that a cone $K$ in a locally convex space (LCS) $X$ is said to be nonoblate if for each neighbourhood of zero $U$, there exists a neighbourhood of zero $V$ for which $V \subset U \cap K - U \cap K$. The theory of differentiation in an LCS as developed in [1] is used systematically. All topological vector spaces considered are assumed to be separated and locally convex.

Let $(G, \tau)$ be a locally convex metric topological vector group (TVG) and $K$ be a closed generating cone in $G$. We will denote by $d$ a quasinorm defining the topology $\tau$, i.e., a nonnegative functional on $G$, which satisfies the conditions:

(a) $0 \leq d(x) \leq 1$ \quad ($x \in G$);

(b) $d(\lambda x) \leq d(x) \quad (|\lambda| \leq 1, x \in G)$;

(c) $d(x_1 + x_2) \leq d(x_1) + d(x_2) \quad (x_1, x_2 \in G)$.

The quasinorm

$$\tilde{d}(x) = \inf \{d(u) + d(v): x = u - v, u, v \in K\},$$

defines on $G$ the topology $\tilde{\tau}$ of a locally convex TVG in which a base of absolutely convex neighbourhoods of zero is formed by the sets

$$V_n = K \cap U_n - K \cap U_n \quad (n = 1, 2, \ldots),$$

where $\{U_n: n = 1, 2, \ldots\}$ is a base of absolutely convex neighbourhoods of zero in the topology $\tau$. It is clear that $\tau \leq \tilde{\tau}$.

Proposition 1. If $(G, \tau)$ is a complete TVG, then it follows from convergence in $(G, \tau)$ of the series:

$$x = \sum_{n=1}^{\infty} x_n,$$

(1)
\[ \tilde{d}(x) \leq \sum_{n=1}^{\infty} \tilde{d}(x_n). \]  

**Proof.** Suppose that the series (1) converges in \((G, \tau)\) and the right-hand side of inequality (2) is finite. Then for every \(\epsilon > 0\), there exist sequences \(u_n \in K\) and \(v_n \in K\) for which \(x_n = u_n - v_n\) and

\[ d(u_n) + d(v_n) \leq \tilde{d}(x_n) + 2^{-n}\epsilon. \]

Since \((G, \tau)\) is a complete TVG, it follows from this that there exist elements \(u, v \in K\) for which \(x = u - v\) and

\[ \tilde{d}(x) \leq d(u) + d(v) \leq \sum_{n=1}^{\infty} [d(u_n) + d(v_n)] \leq \sum_{n=1}^{\infty} \tilde{d}(x_n) + \epsilon. \]

Inequality (2) follows from this since \(\epsilon > 0\) is arbitrary. The proposition is proved.

From Proposition 1 and the completeness of \((G, \tau)\), we deduce that any series (1) for which the right-hand side of inequality (2) is finite converges in \((G, \tau)\). Hence we have

**Proposition 2.** The TVG \((G, \tilde{\tau})\) is complete.

**Proof.** Let \((x_n)\) be a fundamental sequence in \((G, \tilde{\tau})\). We choose a subsequence \((x_{n_k})\) such that

\[ \tilde{d}(x_{n_{k+1}} - x_{n_k}) < 2^{-k} \quad (k = 1, 2\ldots). \]

Then

\[ \sum_{k=1}^{\infty} \tilde{d}(x_{n_{k+1}} - x_{n_k}) < \infty, \]
and consequently, the series

\[ x_{n_1} + \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}), \]

converges in \((G, \tau)\). In the other words, the subsequence \((x_{n_k})\), and along with it also the sequence \((x_n)\), converge in \((G, \tau)\). The proposition is proved.

**Theorem 1.** Let \((X, \tau^*)\) be an SH-space and \(K\) be a generating closed cone in \(X\). Then \(K\) is a nonoblate cone.

**Proof.** Let \((X, \tau^*)\) be an SH-space and \(K\) be a generating closed cone in \(X\). Let

\[ X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} X_{n_1n_2...n_k}, \]

be such that \(\tau^*\) is the strongest locally convex topology on \(X\) for which all the embeddings of the locally convex metric TVGs \(X_{(\nu)}(\nu \in \mathcal{P})\) in the space \((X, \tau^*)\) are continuous. Without loss of generality, it can be assumed that the spaces \(X_{n_1n_2...n_k}(n_k, k = 1, 2, \ldots)\) are seminormed and the embeddings

\[ X_{n_1n_2...n_k+1} \to X_{n_1n_2...n_k} \quad (k = 1, 2, \ldots; \nu \in \mathcal{P}), \]

are continuous. Here \(\mathcal{P}\) is a subset of \(\mathbb{N}^*\), the set of sequences of positive integers.

Let \(\nu = (n_1, n_2, \ldots) \in \mathcal{P}\). We will denote by \(\{U_{n_1n_2...n_k} : k = 1, 2, \ldots\}\) the family of absolutely convex neighbourhoods of zero in a base for the space \(X_{(\nu)}\), which are such that for all \(k = 1, 2, \ldots\), the set \(U_{n_1n_2...n_k}\) is a neighbourhood of zero in \(X_{n_1n_2...n_k}\) and \(2U_{n_1n_2...n_{k+1}} \subseteq U_{n_1n_2...n_k}\). Then the sets
\[ V_{n_1n_2...n_k} = U_{n_1n_2...n_k} \cap K - U_{n_1n_2...n_k} \cap K \quad (k = 1, 2, \ldots), \]

generate absolutely convex and their linear hulls \( L(V_{n_1n_2...n_k}) = Y_{n_1n_2...n_k} \) can be given seminorm topologies in such a way that for each \( k = 1, 2, \ldots \), the sets \( \epsilon V_{n_1n_2...n_k} (\epsilon > 0) \) form a base of neighbourhoods of zero. It is not difficult to see that

\[ X = \bigcup_{v \in \mathcal{P}} \bigcap_{k=1}^{\infty} Y_{n_1n_2...n_k}, \]

and moreover, the sequence \( V_{n_1n_2...n_k} \) forms a base of absolutely convex neighbourhoods of zero for some TVG \( Y_{(v)} \). Since the space \( X_{(v)} \) is complete, then by Proposition 2, the TVG \( Y_{(v)} \) is also complete.

Now, let us consider on \( X \) the strongest locally convex topology \( \sigma^* \) for which all the embeddings of the spaces \( Y_{(v)} (v \in \mathcal{P}) \) in the space \( (X, \sigma^*) \) are continuous. Then \( (X, \sigma^*) \) is an \( SH \)-space and moreover \( \tau^* \leq \sigma^* \). By the Closed Graph Theorem for \( SH \)-spaces, we have the inequality \( \sigma^* \leq \tau^* \). The assertion of the theorem now follows since by construction the cone \( K \) is nonoblate in \( (X, \sigma^*) \). The theorem is proved.

**Corollary 1.** Let \( K \) be a generating closed cone in a sequentially complete bornological \( SH \)-space \( (X, \tau) \). Then \( K \) is a nonoblate cone.

This assertion follows from Theorem 1 and Proposition 7.3.5 of [4].

**2. Compact Differentiability with Respect to a Cone**

Let \( X \) and \( Y \) be LCSs, \( K \) be a closed cone in \( X \) and \( L(X, Y) \) be the vector space of all continuous linear mappings from \( X \) to \( Y \). We will denote by \( \beta \) (resp., \( \beta_c \)) the system of all bounded (resp., compact) subsets
of the space $X$, and by $\beta_k$ (resp., $\beta_k^c$) the system of all bounded (resp., compact) subsets of the cone $K$. Let $L_{\beta}(X, Y)$ (resp., $L_{\beta_k}(X, Y)$) be the LCS obtained by giving the space $L(X, Y)$ the topology of uniform convergence on the sets of the system $\beta$ (resp., $\beta_k$).

We will say (see also [2]) that the operator $A : X \to Y$ is differentiable at the point $x_0 \in X$ in the directions of the cone $K$, if the function $y(t) = A(x_0 + th)$ is differentiable with respect to $t$ at the point $t = 0$ for all $h \in K$. If the derivative $y'(0)$ is representable in the form $y'(0) = A'(x_0)h(h \in K)$, where $A'(x_0) \in L(X, Y)$, then we will call the linear operator $A'(x_0)$ the weak derivative with respect to the cone $K$ at the point $x_0$.

If the identity

$$\lim_{t \to 0} \frac{y(t) - y(0)}{t} = A'(x_0)h,$$

is satisfied uniformly with respect to $h \in B$ for each $B$ from $\beta_k$ (resp., $\beta_k^c$), then we will call $A'(x_0)$ the bounded (resp., compact) derivative with respect to the cone $K$ at the point $x_0$. Mappings which have a weak, bounded or compact derivative with respect to a cone will be called weakly, boundedly or compactly differentiable with respect to the cone.

Let $(X, \tau)$ be a separated sequentially complete bornological $SH$-space, i.e.,

$$X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} X_{n_1n_2\ldots n_k},$$

and each space $X_{(\nu)}$ $(\nu \in \mathcal{P})$ is a locally convex complete metric TVG, which is continuously embedded in $(X, \tau)$. The topology $\tau$ of the space $X$ induces on each space.
\[ X_\nu = \bigcap_{k=1}^{\infty} X_{n_1n_2...n_k}, \]
a locally convex topology \( \tau_\nu \) which in general is different from the topology \( \tau_\nu \) of the Fréchet space \( X_\nu (\nu \in \mathcal{P}) \). We will assume that \( \tau_\nu = \tilde{\tau}_\nu \) for each \( \nu \in \mathcal{P} \).

**Theorem 2.** Suppose that for the operator \( A : X \to Y \) the weak derivative \( A'(x) \) with respect to a generating closed cone \( K \) is a continuous mapping into \( L_{\mathcal{P}_\nu}(X, Y) \) on an open neighbourhood \( U \) of the point \( x \). Then \( A'(x) \) is the compact derivative of the operator \( A \) at the points \( x \in U \).

**Proof.** By Corollary 1, the cone \( K \) is nonoblate in the space \( (X, \tau) \) and we have the identity
\[ X = \bigcup_{\nu \in \mathcal{P}} \bigcap_{k=1}^{\infty} Y_{n_1n_2...n_k}, \]
where the \( Y_{n_1n_2...n_k} \) \((n_k, k = 1, 2, ...)\) are seminormed spaces and the cone \( K \) is nonoblate in each locally convex TVG \( Y(\nu)(\nu \in \mathcal{P}) \). We have to show that
\[ \lim_{\delta \to 0} \frac{A(x + \delta h) - A(x)}{\delta} = A'(x)h, \quad (3) \]
where \( x \in U \) and convergence is uniform with respect to all \( h \in B \) for every \( B \in \beta_c \).

Let \( x \in U, B \in \beta_c \) and let \( W \) be a convex neighbourhood of zero in the space \( Y \). Since the space \( (X, \tau) \) is sequentially complete, the set \( B \) is contained and bounded in some space \( Y_\nu \), where \( \nu \in \mathcal{P} \). By the Closed Graph Theorem, there exists \( \nu' \in \mathcal{P} \) such that \( Y_\nu \subset Y_{\nu'} \). But \( \tau_\nu = \tilde{\tau}_\nu \); therefore, the set \( B \) is compact in \( Y_\nu \) and thus it is compact in \( Y_{\nu'} \). By
Corollary 1 of [3], there is a sequence \((h_n)\) converging to zero in \(Y_{\nu'}\) such that \(B\) is contained in the closed absolutely convex hull of \((h_n)\). Because of the nonoblateness of the cone \(K\) in the space \(Y_{\nu'}\), there exist sequences \((u_n)\subset K\) and \((v_n)\subset K\) for which \(h_n = u_n - v_n\) and \(u_n \to 0, v_n \to 0\) as \(n \to \infty\) in the space \(Y_{\nu'}\). Hence it follows that \(B \subset S - S\), where \(S\) is compact in \((X, \tau)\) and \(S \in \mathbb{P}_c^k\).

Choose \(\delta_0 > 0\) such that \(x + \delta h \in U, x + \delta u \in U,\) and \(x + \delta v \in U,\) where \(|\delta| \leq \delta_0\) and \(h = u - v, h \in B, u, v \in S.\) We introduce the notation

\[
\omega(x, \delta h) = A(x + \delta h) - A(x) - A'(x)\delta h.
\]

It is obvious that

\[
\omega(x, \delta h) = A(x + \delta h) - A(x + \delta h + \delta v) + A(x + \delta h + \delta v) - A(x) - A'(x)\delta h.
\]

Hence by the continuity of \(A'(x)\) at the point \(x,\) we obtain the following identities:

\[
\omega(x, \delta h) = - \int_0^1 A'(x + \delta h + t\delta v)\delta v dt + \int_0^1 A'(x + t(\delta h + \delta v))(\delta h + \delta v)dt
\]

\[
- \int_0^1 A'(x)\delta h dt
\]

\[
= \int_0^1 [A'(x) - A'[x + \delta h + t\delta v]]\delta v dt
\]

\[
+ \int_0^1 [A'(x + t\delta u) - A'(x)]\delta u dt.
\]

Again, by the continuity of \(A'(x)\), there exists a neighbourhood of zero \(P\) in the space \((X, \tau)\) such that for all \(u, v \in S,\) we have the inclusions

\[
[A'(x) - A'(x + P)]v \subset \frac{1}{2} W,
\]
and

\[ [A'(x + P) - A'(x)]u \subset \frac{1}{2} W. \]

Since the set \( S \) is bounded in \((X, \tau)\), there exists \( \delta_W > 0 \) such that for \( \|x\| < \delta_W \), we have (as a result of (4)) the inclusions

\[ \frac{\omega(x, \delta h)}{\delta} \subset \frac{1}{2} W + \frac{1}{2} W = W. \]

Now (3) follows from these inclusions. The theorem is proved.

**Corollary 2.** Let \((X, \tau)\) be the strict inductive limit of the sequence \( \{X_n : n = 1, 2, \ldots\} \) of Fréchet-Montel spaces and let \( K \) be a generating closed cone in \((X, \tau)\). Then if the weak derivative \( A'(x) \) with respect to the cone \( K \) of the operator \( A : X \to Y \) is a continuous mapping into \( L_\beta(X, Y) \) on the open neighbourhood \( U \) of the point \( x \), it is the bounded derivative of the operator \( A \) at the points \( x \in U \).

### 3. The Lagrange Function in SH-Spaces

In this section, we give the application already mentioned of Theorem 1 to the linear optimization problem in an LCS. Suppose that it is required to minimize the functional \( f(x) \) under the condition \( Ax \geq y_0 \), where \( X \) and \( Y \) are LCSs, \( A : X \to Y \) is a continuous linear operator, \( f \) is a continuous linear functional on \( X \); (inequalities in \( Y \) are to be understood in the sense of the ordering defined by the cone \( K \)).

We recall [5] that a point \((x_0, y_0') \in X \times K_{Y'}\) is called a **saddle point** of the Lagrange function

\[ H(x, y') = f(x) - y'(Ax - y_0), \]

if

\[ H(x, y_0') \geq H(x_0, y_0') \geq H(x_0, y') \quad (x \in X, y' \in K_{Y'}). \]

Below we denote by \( Y_0 \) the linear hull in \( Y \) of an element \( y_0 \) and the subspace \( AX \) and by \( M \) the set \( \{x : Ax \geq y_0\} \).
Theorem 3. Let $Y$ be a sequentially complete bornological $SH$-space and let $K_Y$ be a generating closed cone in $Y$; suppose moreover that $Y = Y_0 - K_Y$. Then, the functional $f$ attains a minimum on the set $M$ if and only if the corresponding Lagrange function $H(x, y')$ has a saddle point $(x_0, y'_0) (x \in X, y' \in K_Y)$.

For the proof, it is enough to refer to Corollary 1 and Theorem 9 of [5].

4. Conclusion

Using the closed graph theorem is the important resource for applying of space $Y$. Such condition for space $Y$ is being $SH$-space of Smirnov. In particular, this class contains of Fréchet spaces and spaces $D'(\mathbb{R}^n)$ of generalized functions. So such approach lead to an expansion of mathematical models for economic tasks of optimum control in locally convex spaces.

References


