NEW RESULTS OF STEFFENSEN TYPE INEQUALITIES FOR STIELTJES INTEGRAL

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Abstract

Several generalizations and refinements of some Steffensen type inequalities for Stieltjes integral are established.

1. Introduction

The well-known classical Steffensen’s inequality [6] states:

**Theorem 1.** Let \( f \) and \( g \) be integrable functions defined on \([a, b]\) with \( f \) decreasing, and for each \( t \in [a, b], 0 \leq g(t) \leq 1 \). Then

\[
\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt,
\]

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where \( \lambda = \int_a^b g(t) dt \).

In [1], the Steffensen’s inequality have been generalized in Stieltjes integral type.

Recall that if any one of the two Stieltjes integrals

\[
\int_a^b f(t) dg(t) \quad \text{and} \quad \int_a^b g(t) df(t)
\]

exists, then the other one also exists and one has

\[
\int_a^b f(t) dg(t) + \int_a^b g(t) df(t) = f(b)g(b) - f(a)g(a).
\]

In what follows, we always assume that \( \mu \) is a finite continuous strictly increasing function defined on \([a, b]\). This assures that the inverse \( \mu^{-1} \) exists and is also a finite continuous strictly increasing function defined on \([\mu(a), \mu(b)]\).

If \( f \) is a monotonic function defined on \([a, b]\), then the Stieltjes integral of \( f \) with respect to \( \mu \), i.e.,

\[
\int_a^b f(t) d\mu(t),
\]

clearly exists. For brevity, we would like to agree on saying that \( f \) is \( \mu \)-integrable if and only if \( \int_a^b f(t) d\mu(t) \) exists.

In [1], the following Stieltjes type inequality for Stieltjes integral was proved:

**Theorem 2.** Let \( f \) and \( g \) be \( \mu \)-integrable functions defined on \([a, b]\) with \( f \) decreasing, and for each \( t \in [a, b], 0 \leq g(t) \leq 1 \). Then

\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{\mu^{-1}(\mu(a) + \lambda)} f(t) d\mu(t) \leq \int_a^b f(t) g(t) d\mu(t) \leq \int_a^b f(t) d\mu(t),
\]

(2)

where \( \lambda = \int_a^b g(t) d\mu(t) \).
If we take \( \mu(t) = t \) in Theorem 2, then the inequality (2) reduced to the inequality (1). In [2], a generalization of Theorem 2 was obtained as follows:

**Theorem 3.** Let \( f, g, \) and \( h \) be \( \mu \)-integrable functions defined on \([a, b]\) with \( f \) decreasing, and for each \( t \in [a, b] \), \( 0 \leq g(t) \leq h(t) \). Then

\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t) \leq \int_{a}^{\mu^{-1}(\mu(a) - \lambda)} f(t)h(t)d\mu(t),
\]

(3)

provided that there exists \( \lambda \in [0, \mu(b) - \mu(a)] \) such that

\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t) = \int_{a}^{\mu^{-1}(\mu(a) - \lambda)} h(t)d\mu(t).
\]

If we take \( \mu(t) = t \) in Theorem 3, then the inequality (3) reduced to the corrected version of a Mercer’s result in [4] which has been proved in [3] and [7] in different ways as follows:

**Theorem 4.** Let \( f, g, \) and \( h \) be integrable functions defined on \([a, b]\) with \( f \) decreasing, and for each \( t \in [a, b] \), \( 0 \leq g(t) \leq h(t) \). Then

\[
\int_{a}^{b} f(t)h(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a + \lambda} f(t)h(t)dt,
\]

provided that there exists \( \lambda \in [0, b - a] \) such that

\[
\int_{a}^{a + \lambda} h(t)dt = \int_{a}^{b} g(t)dt = \int_{b - \lambda}^{b} h(t)dt.
\]

**Theorem 5.** Let \( f, g, h, \) and \( k \) be a positive \( \mu \)-integrable functions defined on \([a, b]\) with \( f/k \) decreasing, and \( 0 \leq g \leq h \). Then

\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t) \leq \int_{a}^{\mu^{-1}(\mu(a) - \lambda)} f(t)h(t)d\mu(t),
\]

(4)

provided that there exists \( \lambda \in [0, \mu(b) - \mu(a)] \) such that
Motivated by [7] and [5], in this paper, we will give a further discussion on Steffensen type inequalities for Stieltjes integral. Several generalizations and refinements of some above mentioned inequalities are obtained.

2. Main Results

We first provide a useful lemma.

**Lemma.** Let \( f, g, \) and \( h \) be \( \mu \)-integrable functions defined on \([a, b]\). Suppose also that \( \lambda \in [0, \mu(b) - \mu(a)] \) is a real number such that

\[
\int_a^b h(t) d\mu(t) = \int_a^b g(t) d\mu(t) = \int_a^b h(t) d\mu(t).
\]

Then

\[
\int_a^b f(t) g(t) d\mu(t) = \int_a^b (f(t) h(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)]) d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(b) - \lambda)} h(t) d\mu(t), \tag{5}
\]

and

\[
\int_a^b f(t) g(t) d\mu(t) = \int_a^b (f(t) - f(\mu^{-1}(\mu(b) - \lambda))) [g(t) d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(b) - \lambda)} (f(t) h(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))])
\]

\[
\times [h(t) - g(t)]) d\mu(t). \tag{6}
\]

**Proof.** Observe that \( \lambda \in [0, \mu(b) - \mu(a)] \), we get \( \mu(a) \leq \mu(a) + \lambda \leq \mu(b) \) and \( \mu(a) \leq \mu(b) - \lambda \leq \mu(b) \), which also imply that \( a \leq \mu^{-1}(\mu(a) + \lambda) \leq b \) and \( a \leq \mu^{-1}(\mu(b) - \lambda) \leq b \).
By assumption
\[
\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t) \, d\mu(t) = \int_a^b g(t) \, d\mu(t),
\]
it is not difficult to deduce that
\[
\int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)]) \, d\mu(t)
\]
\[
- \int_a^b f(t)g(t) \, d\mu(t)
\]
\[
= \int_a^{\mu^{-1}(\mu(a)+\lambda)} (f(t)h(t) - f(t)g(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)]) \, d\mu(t)
\]
\[
+ \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)g(t) \, d\mu(t) - \int_a^b f(t)g(t) \, d\mu(t)
\]
\[
= \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(\mu^{-1}(\mu(a) + \lambda))[h(t) - g(t)] \, d\mu(t)
\]
\[
- \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)g(t) \, d\mu(t)
\]
\[
= f(\mu^{-1}(\mu(a) + \lambda)) \left( \int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t) \, d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} g(t) \, d\mu(t) \right)
\]
\[
- \int_a^b f(t)g(t) \, d\mu(t)
\]
\[
= f(\mu^{-1}(\mu(a) + \lambda)) \left( \int_a^b g(t) \, d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} g(t) \, d\mu(t) \right)
\]
\[
- \int_a^b f(t)g(t) \, d\mu(t)
\]
\[
= f(\mu^{-1}(\mu(a) + \lambda)) \int_a^b g(t) \, d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)g(t) \, d\mu(t)
\]
\[
= \int_a^{\mu^{-1}(\mu(a)+\lambda)} \left[ f(\mu^{-1}(\mu(a) + \lambda)) - f(t) \right] g(t) \, d\mu(t),
\]
which is just the desired identity (5) asserted by the Lemma.

Now by assumption

\[
\int_{\mu^1(\mu(b) - \lambda)}^{b} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t),
\]

we can also deduce that

\[
\int_{\mu^1(\mu(b) - \lambda)}^{b} (f(t)h(t) - f(t)g(t) - [f(t) - f(\mu^1(\mu(b) - \lambda))][h(t) - g(t)])d\mu(t)
\]

\[
- \int_{a}^{b} f(t)g(t)d\mu(t)
\]

\[
= \int_{\mu^1(\mu(b) - \lambda)}^{b} f(\mu^1(\mu(b) - \lambda)))[h(t) - g(t)]d\mu(t) - \int_{\mu^1(\mu(b) - \lambda)}^{b} f(t)g(t)d\mu(t)
\]

\[
+ \int_{\mu^1(\mu(b) - \lambda)}^{b} f(t)g(t)d\mu(t) - \int_{a}^{b} f(t)g(t)d\mu(t)
\]

\[
= \int_{\mu^1(\mu(b) - \lambda)}^{b} f(\mu^1(\mu(b) - \lambda)))[h(t) - g(t)]d\mu(t) - \int_{\mu^1(\mu(b) - \lambda)}^{b} f(t)g(t)d\mu(t)
\]

\[
= f(\mu^1(\mu(b) - \lambda))(\int_{\mu^1(\mu(b) - \lambda)}^{b} h(t)d\mu(t) - \int_{\mu^1(\mu(b) - \lambda)}^{b} g(t)d\mu(t))
\]

\[
- \int_{a}^{b} f(t)g(t)d\mu(t)
\]

\[
= f(\mu^1(\mu(b) - \lambda))(\int_{a}^{b} g(t)d\mu(t) - \int_{\mu^1(\mu(b) - \lambda)}^{b} g(t)d\mu(t))
\]

\[
- \int_{a}^{b} f(t)g(t)d\mu(t)
\]

\[
= f(\mu^1(\mu(b) - \lambda))\int_{a}^{b} g(t)d\mu(t) - \int_{\mu^1(\mu(b) - \lambda)}^{b} f(t)g(t)d\mu(t)
\]
\[
\int_{a}^{b} \mu^{-1}(\mu(b) - \lambda) [f(\mu^{-1}(\mu(b) - \lambda)) - f(t)] g(t) d\mu(t),
\]

which is just the desired identity (6) asserted by the Lemma.

**Theorem 6.** Let \( f, g, \) and \( h \) be \( \mu \)-integrable functions defined on \([a, b]\) with \( f \) decreasing, and \( 0 \leq g \leq h \). Suppose also that \( \lambda \in [0, \mu(b) - \mu(a)] \) is a real number such that

\[
\int_{a}^{b} h(t) d\mu(t) = \int_{a}^{b} g(t) d\mu(t) = \int_{a}^{b} h(t) d\mu(t).
\]

Then we have the following inequalities:

\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} f(t) h(t) d\mu(t)
\leq \int_{\mu^{-1}(\mu(b) - \lambda)}^{b} (f(t) h(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))] [h(t) - g(t)]) d\mu(t)
\leq \int_{a}^{b} f(t) g(t) d\mu(t)
\leq \int_{a}^{\mu^{-1}(\mu(a) + \lambda)} (f(t) h(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)]) d\mu(t)
\leq \int_{a}^{\mu^{-1}(\mu(a) + \lambda)} f(t) h(t) d\mu(t). \tag{7}
\]

**Proof.** Since \( f \) is decreasing on \([a, b]\) and \( 0 \leq g \leq h \), we can conclude that

\[
\int_{a}^{\mu^{-1}(\mu(a) + \lambda)} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)] d\mu(t) \geq 0, \tag{8}
\]

\[
\int_{\mu^{-1}(\mu(a) + \lambda)}^{b} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] g(t) d\mu(t) \leq 0, \tag{9}
\]
\[\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][h(t) - g(t))] d\mu(t) \leq 0, \quad (10)\]

and

\[\int_{a}^{\mu^{-1}(\mu(b) - \lambda)} [f(t) - f(\mu^{-1}(\mu(b) - \lambda)))]g(t)d\mu(t) \geq 0. \quad (11)\]

By (5), (8), and (9), we find that

\[\int_{a}^{b} f(t)g(t)d\mu(t)\]

\[\leq \int_{a}^{\mu^{-1}(\mu(a) + \lambda)} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))][h(t) - g(t)]) d\mu(t)\]

\[\leq \int_{a}^{\mu^{-1}(\mu(a) + \lambda)} f(t)h(t)d\mu(t), \quad (12)\]

and by (6), (10), and (11), we get

\[\int_{\mu^{-1}(\mu(b) - \lambda)}^{b} f(t)h(t)d\mu(t)\]

\[\leq \int_{\mu^{-1}(\mu(b) - \lambda)}^{b} (f(t)h(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][h(t) - g(t)]) d\mu(t)\]

\[\leq \int_{a}^{b} f(t)g(t)d\mu(t). \quad (13)\]

Consequently, inequalities (7) follow by combining the inequalities (12) and (13). The proof is completed.

In particular, if we take \( h(t) = 1 \), then we obtain the following refinement of Steffensen type inequality (2).
Corollary 1. Let $f$ and $g$ be $\mu$-integrable functions defined on $[a, b]$ with $f$ decreasing, and for each $t \in [a, b]$, $0 \leq g(t) \leq 1$. Then

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t) d\mu(t)$$

$$\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} (f(t) - [f(t) - f(\mu^{-1}(\mu(b) - \lambda))][1 - g(t)])d\mu(t)$$

$$\leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} (f(t) - [f(t) - f(\mu^{-1}(\mu(a) + \lambda))][1 - g(t)])d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)d\mu(t), \quad (14)$$

where $\lambda = \int_{a}^{b} g(t)d\mu(t)$.

Theorem 7. Let $f$, $g$, $h$, and $\psi$ be $\mu$-integrable functions defined on $[a, b]$ with $f$ decreasing, and $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$ for $t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that

$$\int_{a}^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_{a}^{b} g(t) d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t).$$

Then we have the following inequalities:

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) + \int_{a}^{b} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))]\psi(t)d\mu(t)$$

$$\leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

$$\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t) - \int_{a}^{b} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))]\psi(t)d\mu(t). \quad (15)$$
Proof. Since $f$ is decreasing on $[a, b]$ and $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$ for $t \in [a, b]$, we have

\[
\int_a^{\mu^{-1}(\mu(a)+\lambda)} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)] d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(a)+\lambda)}^{b} [f(\mu^{-1}(\mu(a) + \lambda)) - f(t)] g(t) d\mu(t)
\]

\[
= \int_a^{\mu^{-1}(\mu(a)+\lambda)} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] [h(t) - g(t)] d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(a)+\lambda)}^{b} [f(\mu^{-1}(\mu(a) + \lambda)) - f(t)] g(t) d\mu(t)
\]

\[
\geq \int_a^{\mu^{-1}(\mu(a)+\lambda)} [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] \psi(t) d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(a)+\lambda)}^{b} [f(\mu^{-1}(\mu(a) + \lambda)) - f(t)] \psi(t) d\mu(t) d\mu(t)
\]

\[
= \int_a^{b} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))] \psi(t) d\mu(t), \tag{16}
\]

and

\[
\int_a^{\mu^{-1}(\mu(b)-\lambda)} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))] g(t) d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} [f(\mu^{-1}(\mu(b) - \lambda)) - f(t)] [h(t) - g(t)] d\mu(t)
\]

\[
= \int_a^{\mu^{-1}(\mu(b)-\lambda)} [f(t) - f(\mu^{-1}(\mu(b) - \lambda))] g(t) d\mu(t)
\]

\[
+ \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} [f(\mu^{-1}(\mu(b) - \lambda)) - f(t)] [h(t) - g(t)] d\mu(t)
\]
By combining the identities (5), (6) and inequalities (16), (17), we get the inequality (15) asserted by Theorem 7. Thus the proof is completed.

Corollary 2. Let $f$ and $g$ be $\mu$-integrable functions defined on $[a, b]$ with $f$ decreasing, and $0 \leq M \leq g(t) \leq 1 - M$ for $t \in [a, b]$. Then

$$
\int_a^b f(t) d\mu(t) + M \int_a^b [f(t) - f(\mu^{-1}(\mu(b) - \lambda))] d\mu(t)
\leq \int_a^b f(t) g(t) d\mu(t)
\leq \int_a^b f(t) d\mu(t) - M \int_a^b [f(t) - f(\mu^{-1}(\mu(a) + \lambda))] d\mu(t),
$$

where $\lambda = \int_a^b g(t) d\mu(t)$.

Remark 1. Clearly, the inequality (18) is a refinement and generalization of Steffensen type inequality (2). Indeed, in its special case when $M = 0$, the inequality (18) would reduce to Steffensen type inequality (2).

Now, we would like to give a general result on a considerably improved version of Steffensen type inequality (2) by introducing the additional parameters $\lambda_1$ and $\lambda_2$. 
Theorem 8. Let $f$ and $g$ be $\mu$-integrable functions defined on $[a, b]$ with $f$ decreasing, and $0 \leq M \leq g(t) \leq 1 - M$ for $t \in [a, b]$. Also let $0 \leq \lambda_1 \leq \int_a^b g(t) d\mu(t) \leq \lambda_2 \leq \mu(b) - \mu(a)$. Then we have the following inequalities:

\[
\int_{\mu^{-1}(\mu(b) - \lambda_1)}^b f(t) d\mu(t) + f(b) \left( \int_a^b g(t) d\mu(t) - \lambda_1 \right) + M \int_a^b |f(t) - f(\mu^{-1}(\mu(b) - \int_a^b g(t) d\mu(t)))| d\mu(t) \leq \int_a^b f(t) g(t) d\mu(t) \leq \int_{\mu^{-1}(\mu(a) + \lambda_2)}^a f(t) d\mu(t) - f(b) (\lambda_2 - \int_a^b g(t) d\mu(t)) - M \int_a^b |f(t) - f(\mu^{-1}(\mu(a) + \int_a^b g(t) d\mu(t)))| d\mu(t). \tag{19}
\]

Proof. By assumption, it is clear that

\[
\mu(a) \leq \mu(a) + \lambda_1 \leq \mu(a) + \int_a^b g(t) d\mu(t) \leq \mu(a) + \lambda_2 \leq \mu(b),
\]

and

\[
\mu(a) \leq \mu(b) - \lambda_2 \leq \mu(b) - \int_a^b g(t) d\mu(t) \leq \mu(b) - \lambda_1 \leq \mu(b),
\]

which also imply that

\[
a \leq \mu^{-1}(\mu(a) + \lambda_1) \leq \mu^{-1}(\mu(a) + \int_a^b g(t) d\mu(t)) \leq \mu^{-1}(\mu(a) + \lambda_2) \leq b,
\]

and

\[
a \leq \mu^{-1}(\mu(b) - \lambda_2) \leq \mu^{-1}(\mu(b) - \int_a^b g(t) d\mu(t)) \leq \mu^{-1}(\mu(b) - \lambda_1) \leq b.
\]
By direct computation, we get
\[
\int_a^b f(t)g(t)d\mu(t) - \int_a^b \mu^{-1}(\mu(a) + \lambda_2) f(t)d\mu(t) + f(b)(\lambda_2 - \int_a^b g(t)d\mu(t))
\]
\[
= \int_a^b f(t)g(t)d\mu(t) - \int_a^b \mu^{-1}(\mu(a) + \lambda_2) f(t)d\mu(t) + \int_a^b \mu^{-1}(\mu(a) + \lambda_2) f(b)d\mu(t)
\]
\[
- \int_a^b f(b)g(t)d\mu(t)
\]
\[
= \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^b \mu^{-1}(\mu(a) + \lambda_2) [f(t) - f(b)]d\mu(t)
\]
\[
\leq \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^b \mu^{-1}(\mu(a) + \lambda_2) \int_a^b g(t)d\mu(t) [f(t) - f(b)]d\mu(t),
\]
where the last inequality follows from the assumption that
\[
a \leq \mu^{-1}(\mu(a) + \int_a^b g(t)d\mu(t)) \leq \mu^{-1}(\mu(a) + \lambda_2) \leq b,
\]
and
\[
f(t) - f(b) \geq 0 \text{ for } t \in [a, b].
\]

On the other hand, since the function \(f(t) - f(b)\) is \(\mu\)-integrable and decreasing on \([a, b]\), thus by using Corollary 2 with the following substitution:
\[
f(t) \mapsto f(t) - f(b),
\]
in (18), we find that
\[
\int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_a^b \mu^{-1}(\mu(a) + \lambda_2) \int_a^b g(t)d\mu(t) [f(t) - f(b)]d\mu(t)
\]
\[
\leq -M \int_a^b [f(t) - f(b) - f(\mu^{-1}(\mu(a) + \int_a^b g(t)d\mu(t)))d\mu(t)]d\mu(t).\]
By combining the inequalities (20) and (21), we obtain
\[
\int_a^b f(t)g(t)d\mu(t) - \int_a^b f(t)d\mu(t) + f(b)(\lambda_2 - \int_a^b g(t)d\mu(t))
\leq -M\int_a^b |f(t) - f(b) - f(\mu^{-1}(\mu(a) + \int_a^b g(t)d\mu(t))))|d\mu(t),
\]
which is the second inequality in the assertion (19) of Theorem 8.

Similarly, we can also prove that
\[
\int_a^b f(t)g(t)d\mu(t) - \int_a^b f(t)d\mu(t) - f(b)(\int_a^b g(t)d\mu(t) - \lambda_1)
\geq \int_a^b [f(t) - f(b)]g(t)d\mu(t) - \int_{\mu^{-1}(\mu(b)-\lambda_1)}^b \int_a^b g(t)d\mu(t))g(t)d\mu(t)
\geq M\int_a^b |f(t) - f(\mu^{-1}(\mu(b) - \int_a^b g(t)d\mu(t))))|d\mu(t),
\]
which implies the first inequality in the assertion (19) of Theorem 8. Thus completes the proof of Theorem 8.

**Remark 2.** It is clear that Steffensen type inequality (2) would follow as a special case of the inequality (19) when
\[M = 0 \quad \lambda_1 = \lambda_2.\]

Moreover, it is worth noticing that the inequality (19) is stronger than Steffensen type inequality (2) if \(f(b) \geq 0\).

**Theorem 9.** Let \(h\) be a positive \(\mu\)-integrable functions on \([a, b]\) and \(f, g\) be \(\mu\)-integrable functions on \([a, b]\) such that \(\frac{f}{h}\) is decreasing, and \(0 \leq g(t) \leq 1, \ t \in [a, b]\). Suppose also that \(\lambda \in [0, \mu(b) - \mu(a)]\) is a real number such that
\[
\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b h(t)g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^b h(t)d\mu(t).
\]
Then we have the following inequalities:

$$
\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t) \, d\mu(t)
\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} \left( f(t) - \frac{f(t) - f(\mu^{-1}(\mu(b) - \lambda))}{h(\mu^{-1}(\mu(b) - \lambda))} \right) h(t) \left[ 1 - g(t) \right] \, d\mu(t)
\leq \int_{a}^{b} f(t) g(t) \, d\mu(t)
$$

$$
\leq \int_{a}^{b} \left( f(t) - \frac{f(t) - f(\mu^{-1}(\mu(a) + \lambda))}{h(\mu^{-1}(\mu(a) + \lambda))} \right) h(t) \left[ 1 - g(t) \right] \, d\mu(t)
\leq \int_{a}^{b} \frac{f(t) - f(t)}{h(t)} \, d\mu(t).
$$

**Proof.** Take the substitutions $f(t) \mapsto \frac{f(t)}{h(t)}$ and $g(t) \mapsto h(t)g(t)$ in Theorem 6.

**Theorem 10.** Let $k$ be a positive $\mu$-integrable functions on $[a, b]$ and $f, g, h$ be $\mu$-integrable functions on $[a, b]$ such that $\frac{f(t)}{k(t)}$ is decreasing, and $0 \leq g(t) \leq h(t)$, $t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that

$$
\int_{a}^{b} h(t) k(t) \, d\mu(t) = \int_{a}^{b} g(t) k(t) \, d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t) k(t) \, d\mu(t).
$$

Then we have the following inequalities:

$$
\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t) h(t) \, d\mu(t)
\leq \int_{\mu^{-1}(\mu(b)-\lambda)}^{b} \left( f(t) h(t) - \frac{f(t) - f(\mu^{-1}(\mu(b) - \lambda))}{k(\mu^{-1}(\mu(b) - \lambda))} \right) \, d\mu(t).
$$
≤ \int_a^b f(t)g(t)\,d\mu(t)

≤ \int_a^b \mu^{-1}(\mu(a)+\lambda) f(t)h(t) - \left[ \frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{k(\mu^{-1}(\mu(a)+\lambda))} \right]k(t)[h(t) - g(t)]d\mu(t)

≤ \int_a^b \mu^{-1}(\mu(a)+\lambda) f(t)h(t)\,d\mu(t).

**Proof.** Take the substitutions \( f(t) \mapsto \frac{f(t)}{k(t)}, g(t) \mapsto k(t)g(t), \) and \( h(t) \mapsto k(t)h(t) \) in Theorem 6.

**Theorem 11.** Let \( h \) be a positive \( \mu \)-integrable functions on \([a, b]\) and \( f, g, \psi \) be \( \mu \)-integrable functions on \([a, b]\) such that \( \frac{f}{h} \) is decreasing, and \( 0 \leq \psi(t) \leq g(t) \leq 1 - \psi(t) \) for \( t \in [a, b] \). Suppose also that \( \lambda \in [0, \mu(b) - \mu(a)] \) is a real number such that

\[
\int_a^b \mu^{-1}(\mu(a)+\lambda) h(t)\,d\mu(t) = \int_a^b h(t)g(t)\,d\mu(t) = \int_{\mu^{-1}(\mu(b)-\lambda)} h(t)\,d\mu(t).
\]

Then we have the following inequalities:

\[
\int_{\mu^{-1}(\mu(b)-\lambda)}^b f(t)\,d\mu(t) + \int_a^b \left[ \frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(b)-\lambda))}{h(\mu^{-1}(\mu(b)-\lambda))} \right]h(t)\psi(t)\,d\mu(t)
\]

≤ \int_a^b f(t)g(t)\,d\mu(t)

≤ \int_a^b \mu^{-1}(\mu(a)+\lambda) f(t)\,d\mu(t) - \int_a^b \left[ \frac{f(t)}{h(t)} - \frac{f(\mu^{-1}(\mu(a)+\lambda))}{h(\mu^{-1}(\mu(a)+\lambda))} \right]h(t)\psi(t)\,d\mu(t).

**Proof.** Take the substitutions \( f(t) \mapsto \frac{f(t)}{h(t)}, g(t) \mapsto h(t)g(t) \) and \( \psi(t) \mapsto h(t)\psi(t) \) in Theorem 7.
Theorem 12. Let $k$ be a positive $\mu$-integrable functions on $[a, b]$ and $f, g, h, \psi$ be $\mu$-integrable functions on $[a, b]$ such that $\frac{f}{k}$ is decreasing, and $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$ for $t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that
\[
\int_a^b h(t)k(t)d\mu(t) = \int_a^b g(t)k(t)d\mu(t) = \int_{\mu^{-1}(\mu(b) - \lambda)}^{\mu^{-1}(\mu(a) - \lambda)} h(t)k(t)d\mu(t).
\]
Then we have the following inequalities:
\[
\int_{\mu^{-1}(\mu(b) - \lambda)}^{\mu^{-1}(\mu(a) + \lambda)} f(t)h(t)d\mu(t) + \int_a^b \left[ \frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(b) - \lambda))}{k(\mu^{-1}(\mu(a) - \lambda))} \right] k(t)\psi(t)d\mu(t)
\leq \int_a^b f(t)g(t)d\mu(t)
\leq \int_a^b f(t)h(t)d\mu(t) - \int_a^b \left[ \frac{f(t)}{k(t)} - \frac{f(\mu^{-1}(\mu(a) + \lambda))}{k(\mu^{-1}(\mu(a) + \lambda))} \right] k(t)\psi(t)d\mu(t).
\]
(22)

Proof. Take the substitutions $f(t) \mapsto \frac{f(t)}{k(t)}$, $g(t) \mapsto k(t)g(t)$, $h(t) \mapsto k(t)h(t)$ and $\psi(t) \mapsto k(t)\psi(t)$ in Theorem 7.

Remark 3. Clearly, the inequality (22) is a refinement and generalization of Steffensen type inequality (4).

Corollary 3. Let $f, g, h, \psi$ be $\mu$-integrable functions on $[a, b]$ with $f$ is decreasing, and $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$ for $t \in [a, b]$. Suppose also that $\lambda \in [0, \mu(b) - \mu(a)]$ is a real number such that
\[
\int_a^b h(t)d\mu(t) = \int_a^b g(t)d\mu(t) = \int_{\mu^{-1}(\mu(b) - \lambda)}^{\mu^{-1}(\mu(a) + \lambda)} h(t)d\mu(t).
\]
Then we have the following inequalities:

\[
\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)\,d\mu(t) + \int_{a}^{b} \left[ \int_{a}^{b} g(t)\,d\mu(t) \right] \varphi(t)\,d\mu(t)
\]

\[
\leq \int_{a}^{b} f(t)g(t)\,d\mu(t)
\]

\[
\leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)\,d\mu(t) - \int_{a}^{b} \left[ \int_{a}^{b} g(t)\,d\mu(t) \right] \varphi(t)\,d\mu(t). \quad (23)
\]

**Remark 4.** Clearly, the inequality (23) is a refinement and generalization of Steffensen type inequality (3).

**References**


