INEQUALITIES ON THE LINE SEGMENT

ZLATKO PAVIĆ

Mechanical Engineering Faculty in Slavonski Brod
University of Osijek
Trg Ivane Brlić Mažuranić 2
35000 Slavonski Brod
Croatia
e-mail: Zlatko.Pavic@sfsb.hr

Abstract

The article deals with generalizations of the inequalities for convex functions on the line segment. The Jensen and the Hermite-Hadamard inequalities are included in the study. Some improvements of the Hermite-Hadamard inequality are obtained and applied to mathematical means.

1. Introduction

Let $X$ be a real linear space. A linear combination $a\alpha + b\beta$ of points $a, b \in X$ and coefficients $\alpha, \beta \in \mathbb{R}$ is affine if $\alpha + \beta = 1$. A set $S \subseteq X$ is affine if it contains all binomial affine combinations of its points. A function $h : S \to \mathbb{R}$ is affine if the equality

$$h(\alpha a + \beta b) = \alpha h(a) + \beta h(b),$$

holds for every binomial affine combination $a\alpha + b\beta$ of the affine set $S$. 
Convex combinations and sets are introduced by restricting to affine combinations with nonnegative coefficients. A function \( h : S \to \mathbb{R} \) is convex if the inequality
\[
f(aa + \beta b) \leq \alpha f(a) + \beta f(b),
\]
holds for every binomial convex combination \( aa + \beta b \) of the convex set \( S \).

Using mathematical induction, the above concept can be extended to \( n \)-membered affine or convex combinations.

In this paper, we use the real line \( \mathbb{R} = \mathbb{R} \). Besides convex and affine combinations, we will use barycenters of the sets of real numbers. If \( \mu \) is a positive measure on \( \mathbb{R} \), and if \( S \subseteq \mathbb{R} \) is a measurable set such that \( \mu(S) > 0 \), then the integral mean point
\[
c = \frac{1}{\mu(S)} \int_S x \, d\mu,
\]
is called the barycenter of the set \( S \) respecting the measure \( \mu \), or just the set barycenter. The barycenter \( c \) belongs to the convex hull of the set \( S \), as the smallest convex set containing \( S \). Given the measurable set \( S \) of positive measure, every affine function \( h : \mathbb{R} \to \mathbb{R} \) satisfies the equality
\[
h \left( \frac{1}{\mu(S)} \int_S x \, d\mu \right) = \frac{1}{\mu(S)} \int_S h(x) \, d\mu.
\]
For the purpose of the paper, the set \( S \) will be used as an interval or a union of intervals.

2. The Jensen and the Hermite-Hadamard Inequalities

Through the paper, we will use a bounded interval of real numbers with endpoints \( a < b \). Each point \( c \in [a, b] \) can be presented by the unique binomial convex combination
\[ c = aa + \beta b, \]  
\[ \text{where} \]  
\[ \alpha = \frac{b - c}{b - a}, \quad \beta = \frac{c - a}{b - a}. \]  

The next two lemmas present the properties of a convex function \( f : [a, b] \to \mathbb{R} \) concerning its supporting and secant line.

The discrete version refers to interval points and interval endpoints sharing the common center.

**Lemma A.** Let \([a, b]\) be a closed interval of real numbers, and let \( \sum_{i=1}^{n} \lambda_i x_i \) be a convex combination of points \( x_i \in [a, b] \). Let \( aa + \beta b \) be the unique endpoints convex combination such that

\[ \sum_{i=1}^{n} \lambda_i x_i = aa + \beta b. \]  

Then every convex function \( f : [a, b] \to \mathbb{R} \) satisfies the double inequality

\[ f(aa + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \leq af(a) + \beta f(b). \]  

**Proof.** Taking \( c = \sum_{i=1}^{n} \lambda_i x_i \), we have the following two cases.

If \( c \in \{a, b\} \), then Equation (8) is reduced to \( f(c) \leq f(c) \leq f(c) \).

If \( c \in (a, b) \), then using a supporting line \( y = h_1(x) \) of the convex curve \( y = f(x) \) at the graph point \( C(c, f(c)) \), and the secant line \( y = h_2(x) \) passing through the graph points \( A(a, f(a)) \) and \( B(b, f(b)) \), we get the inequality.
\[ f(aa + \beta b) = h_1(aa + \beta b) = \sum_{i=1}^{n} \lambda_i h_1(x_i) \]

\[ \leq \sum_{i=1}^{n} \lambda_i f(x_i) \]

\[ \leq \sum_{i=1}^{n} \lambda_i h_2(x_i) = h_2(aa + \beta b) = af(a) + \beta f(b), \] (9)

containing Equation (8).

The discrete-integral version refers to the connection of the interval barycenter with interval endpoints.

**Lemma B.** Let \([a, b]\) be a closed interval of real numbers, and let \(\mu\) be a positive measure on \(\mathbb{R}\) such that \(\mu([a, b]) > 0\). Let \(aa + \beta b\) be the unique endpoints convex combination such that

\[ \frac{1}{\mu([a, b])} \int_{[a, b]} x \, d\mu = aa + \beta b. \] (10)

Then every convex function \(f : [a, b] \to \mathbb{R}\) satisfies the double inequality

\[ f(aa + \beta b) \leq \frac{1}{\mu([a, b])} \int_{[a, b]} f(x) \, d\mu \leq af(a) + \beta f(b). \] (11)

**Proof.** The proof can be done utilizing Equation (9) so that the integral means are used instead of the \(n\)-membered convex combinations.

We emphasize the basic content of Lemma A. Using the left-hand side of the inequality in Equation (8) with the \(n\)-membered convex combination instead of the binomial endpoints convex combination, we obtain the discrete form of Jensen’s inequality

\[ f \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i f(x_i). \] (12)

Using the Riemann integral in Lemma B, the condition in (10) gives the midpoint
\[
\frac{1}{b-a} \int_a^b x \, dx = \frac{1}{2} a + \frac{1}{2} b, \tag{13}
\]
and its use in Equation (11) implies the classic Hermite-Hadamard inequality
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{14}
\]
Moreover, the inequality in Equation (14) follows by integrating the supporting-secant line inequality
\[
h_1(x) \leq f(x) \leq h_2(x), \tag{15}
\]
over the interval \([a, b]\).

We finish the section with a historic note on these two important inequalities. In 1905, applying the inductive principle, Jensen (see [4]) extended the inequality in Equation (2) to \(n\)-membered convex combinations. In 1906, working on transition to integrals, Jensen (see [5]) stated the another form. In 1883, studying convex functions, Hermite (see [3]) attained the inequality in Equation (14). In 1893, not knowing Hermite’s result, Hadamard (see [2]) got the left-hand side of Equation (14). For information as regards the Jensen and the Hermite-Hadamard inequalities, one may refer to papers [1], [6], [9], [10], [11], and [12].

3. Main Results

To refine the Hermite-Hadamard inequality in Equation (14), we will use convex combinations of points of the closed interval \([a, b]\). In the main Theorem 3.1, we improve Equation (14) by using convex combinations of the midpoint \((a + b)/2\). The concluding Theorem 3.4 presents the integral refinement of Equation (14).

We take points \(c, d \in [a, b]\) such that
\[
\frac{c + d}{2} = \frac{a + b}{2}. \tag{16}
\]
Applying the right-hand side of the inequality in Equation (8) to the above assumption, and multiplying by 2, we obtain the simple inequality

\[ f(c) + f(d) \leq f(a) + f(b), \]

that will be used in this section. The main theorem follows.

**Theorem 3.1.** Let \([a, b]\) be a closed interval of real numbers, and let \([c, d] \subset (a, b)\) be a closed subinterval satisfying the common barycenter condition in Equation (16).

Then every convex function \(f : [a, b] \to \mathbb{R}\) satisfies the series of inequalities

\[
\begin{align*}
f\left(\frac{a + b}{2}\right) & \leq \frac{1}{4} f\left(\frac{a + c}{2}\right) + \frac{1}{2} f\left(\frac{c + d}{2}\right) + \frac{1}{4} f\left(\frac{d + b}{2}\right) \\
& \leq \frac{1}{4} \int_a^c f(x)dx + \frac{1}{2} \int_c^d f(x)dx + \frac{1}{4} \int_d^b f(x)dx \\
& \leq \frac{f(a) + 3f(c) + 3f(d) + f(b)}{8} \leq \frac{f(a) + f(b)}{2}.
\end{align*}
\]

**Proof.** Applying the Hermite-Hadamard inequality to the convex combination of points \(c\) and \(d\) written as

\[ t = \frac{1}{2} c + \frac{1}{2} d, \]

we have

\[
\begin{align*}
f(t) &= f\left(\frac{c + d}{2}\right) \\
& \leq \int_c^d f(x)dx \\
& \leq \frac{f(c) + f(d)}{2}.
\end{align*}
\]
Applying the same procedure to the convex combination of midpoints 
\((a + c) / 2\) and \((d + b) / 2\) given as

\[
t = \frac{1}{2} a + c + \frac{1}{2} d + b.
\]  

we get

\[
f(t) \leq \frac{1}{2} f\left(\frac{a + c}{2}\right) + \frac{1}{2} f\left(\frac{d + b}{2}\right)
\]

\[
\leq \frac{1}{2} \int_{a}^{c} f(x)dx + \frac{1}{2} \int_{d}^{b} f(x)dx
\]

\[
\leq \frac{1}{4} f(a) + f(c) + f(d) + f(b).
\]

Taking the arithmetic means of the inequalities in Equations (20) and (22), using Equation (17) and rearranging, we obtain the inequality in Equation (18).

The inequality in Equation (18) can be expressed using the point 
\(d = a + b - c\). The observed Equation (18) can also be expressed with the point \(c = a + 2\delta\), where \(0 < \delta < (b - a) / 2\). Using this choice, we have 
\(d = b - 2\delta\), \((a + c) / 2 = a + \delta\) and \((d + b) / 2 = b - \delta\). Finally, we can use the convex combinations

\[
c = \alpha a + \beta b, \quad d = (1 - \alpha)a + (1 - \beta)b,
\]

provided that \(\alpha a + \beta b < (a + b) / 2\). Regardless of all these cases, the midpoint \((a + b) / 2\) is not covered on the right side of Equation (18).

The inequality in Equation (18) does not include the case \(c = d = (a + b) / 2\). A method similar to that in Theorem 3.1 can be applied to intervals \([a, (a + b) / 2]\) and \(((a + b) / 2, b]\) and so derive the inequality

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2} \left[f\left(\frac{3a + b}{4}\right) + f\left(\frac{a + 3b}{4}\right)\right].
\]
The above improvement of the Hermite-Hadamard inequality was noted in [8].

A convex function \( f : [a, b] \rightarrow \mathbb{R} \) satisfies the inequality

\[
f(a + b - c) \leq f(a) + f(b) - f(c),
\]

for every point \( c \in [a, b] \) by Equation (17). The above simple inequality can be generalized by using the convex combination \( \sum_{i=1}^{n} \gamma_i c_i \) instead of the point \( c \). Applying Jensen's inequality to the convex combination

\[
t = a + b - \sum_{i=1}^{n} \gamma_i c_i = \sum_{i=1}^{n} \gamma_i (a + b - c_i),
\]

and using Equation (25), Mercer (see [7]) obtained the inequality

\[
f \left( a + b - \sum_{i=1}^{n} \gamma_i c_i \right) \leq f(a) + f(b) - \sum_{i=1}^{n} \gamma_i f(c_i).
\]

**Corollary 3.2.** Let \( [a, b] \) be a closed interval of real numbers, let \( c \in (a, b) \) be an open interval point, and let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function.

**If** \( c \leq (a + b)/2 \), **then**

\[
f(a + b - c) + f(c) \leq \frac{\int_{a}^{c} f(x)dx + \int_{a+b-c}^{c} f(x)dx}{c - a} \leq f(a) + f(b).
\]

**If** \( c \geq (a + b)/2 \), **then**

\[
f(a + b - c) + f(c) \leq \frac{\int_{a+b-c}^{a} f(x)dx + \int_{c}^{b} f(x)dx}{c - a} \leq f(a) + f(b).
\]
Proof. We prove the double inequality in Equation (28). Let \( y = h(x) \) be the secant line of the convex curve \( y = f(x) \) passing through the graph points \( C(c, f(c)) \) and \( D(a + b - c, f(a + b - c)) \). Using the affinity of the secant function \( h \) specified in Equations (1) and (4), and the inequality \( h(x) \leq f(x) \) for \( x \in [a, c] \cup [a + b - c, b] \), we get

\[
\frac{1}{2} f(c) + \frac{1}{2} f(a + b - c) = \frac{1}{2} h(c) + \frac{1}{2} h(a + b - c)
\]

\[
= h\left(\frac{a + b}{2}\right) = h\left(\frac{1}{2} \int_a^c x \, dx + \frac{1}{2} \int_{a+b-c}^b x \, dx \right)
\]

\[
= \frac{1}{2} \int_a^c h(x) \, dx + \frac{1}{2} \int_{a+b-c}^b h(x) \, dx \leq \frac{1}{2} \int_a^c f(x) \, dx + \frac{1}{2} \int_{a+b-c}^b f(x) \, dx
\]

Continuing with Equation (30) by applying the Hermite-Hadamard inequality and the inequality in Equation (25), it follows

\[
\leq \frac{f(a) + f(c) + f(a + b - c) + f(b)}{4} \leq \frac{f(a) + f(b)}{2}.
\]

It remains only to multiply by 2.

Corollary 3.3. Let \([a, b]\) be a closed interval of real numbers, and let \([c, d]\) be a closed subinterval satisfying the common barycenter condition in Equation (16).

Then every convex function \( f : [a, b] \to \mathbb{R} \) satisfies the double integral inequality

\[
\frac{\int_b^d f(x) \, dx}{d - c} \leq \frac{\int_a^b f(x) \, dx}{b - a} \leq \frac{\int_a^c f(x) \, dx}{2(c - a)} + \frac{\int_{a+b-c}^b f(x) \, dx}{2(b - d)}.
\]
Proof. Midpoint equality in Equation (16) can be expressed in the integral form by the barycenter equalities

\[
\frac{\int_c^d x \, dx}{d - c} = \frac{\int_a^b x \, dx}{b - a} = \frac{\int_a^c x \, dx}{2(c - a)} + \frac{\int_c^b x \, dx}{2(b - d)}.
\]

We firstly prove that the left term of Equation (32) is less than or equal to the right term. Let \( y = h(x) \) be the secant line of the convex curve \( y = f(x) \) passing through the graph points \( C(c, f(c)) \) and \( D(d, f(d)) \). Using inequalities \( f(x) \leq h(x) \) for \( x \in [c, d] \), and \( h(x) \leq f(x) \) for \( x \in [a, c] \cup [a + b - c, d] \), and applying the affinity of function \( h \) to Equation (33), we get

\[
\frac{\int_c^d f(x) \, dx}{d - c} \leq \frac{\int_c^d h(x) \, dx}{d - c} = \frac{\int_a^c h(x) \, dx}{2(c - a)} + \frac{\int_c^b h(x) \, dx}{2(b - d)}
\]

\[
\leq \frac{\int_a^c f(x) \, dx}{2(c - a)} + \frac{\int_d^b f(x) \, dx}{2(b - d)}.
\]

Now, the inequality in Equation (32) can be confirmed by the combination

\[
\frac{\int_a^b f(x) \, dx}{b - a} = \frac{\int_a^c f(x) \, dx + \int_c^d f(x) \, dx + \int_d^b f(x) \, dx}{b - a} = \alpha \left[ \frac{\int_c^d f(x) \, dx}{d - c} \right] + \beta \left[ \frac{\int_a^c f(x) \, dx}{2(c - a)} + \frac{\int_d^b f(x) \, dx}{2(b - d)} \right],
\]

which is convex because the coefficients

\[
\alpha = \frac{d - c}{b - a}, \quad \beta = \frac{2(c - a)}{b - a} = \frac{2(d - b)}{b - a}
\]

are nonnegative and their sum is equal to 1. \( \square \)
Combining the inequalities in Equations (14), (32), (28), and (29), we get the following integral refinement of the Hermite-Hadamard inequality.

**Theorem 3.4.** Let \([a, b]\) be a closed interval of real numbers, and let \([c, d] \subset (a, b)\) be a closed subinterval satisfying the common barycenter condition in Equation (16). Then every convex function \(f: [a, b] \to \mathbb{R}\) satisfies the series of inequalities

\[
f\left(\frac{a + b}{2}\right) \leq \frac{\int_c^d f(x)dx}{d - c} \leq \frac{\int_a^b f(x)dx}{b - a} \leq \frac{\int_a^c f(x)dx}{2(c - a)} + \frac{\int_d^b f(x)dx}{2(b - d)} \leq \frac{f(a) + f(b)}{2}.
\]

(37)

4. Application to Means

Throughout this section we use positive numbers \(a\) and \(b\), and a strictly monotone continuous function \(\varphi: [a, b] \to \mathbb{R}\).

The discrete quasi-arithmetic mean of the numbers \(a\) and \(b\) respecting the function \(\varphi\) can be defined by the number

\[
M_\varphi(a, b) = \varphi^{-1}\left(\frac{1}{2} \varphi(a) + \frac{1}{2} \varphi(b)\right).
\]

(38)

Using the identity function \(\varphi(x) = x\), we get the generalized arithmetic mean

\[
A(a, b) = \frac{1}{2} a + \frac{1}{2} b,
\]

(39)

using the hyperbolic function \(\varphi(x) = 1/x\), we have the generalized harmonic mean

\[
H(a, b) = \left(\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}\right)^{-1},
\]

(40)
and using the logarithmic function \( \varphi(x) = \ln x \), we obtain the generalized geometric mean

\[
G(a, b) = a^{\frac{1}{2}} b^{\frac{1}{2}}.
\]

The above means satisfy the generalized harmonic-geometric-arithmetic mean inequality

\[
H(a, b) < G(a, b) < A(a, b).
\]

The integral quasi-arithmetic mean of the numbers \( a \) and \( b \) respecting the function \( \varphi \) can be defined by the number

\[
M_\varphi(a, b) = \varphi^{-1}\left( \frac{1}{b-a} \int_a^b \varphi(x) dx \right).
\]

Using the hyperbolic function, we have the logarithmic mean

\[
L(a, b) = \left( \frac{1}{b-a} \int_a^b \frac{1}{x} dx \right)^{-1} = \frac{b-a}{\ln b - \ln a},
\]

and using the logarithmic function, we obtain the identric mean

\[
I(a, b) = \exp\left( \frac{1}{b-a} \int_a^b \ln x \, dx \right) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}.
\]

The well-known mean inequality asserts that

\[
H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b).
\]

Applying Equation (18) to the convex function \( f(x) = -\ln x \) using substitutions \( a \mapsto 1/a, b \mapsto 1/b, c \mapsto 1/c, \) and \( d \mapsto 1/d, \) and then acting on the rearranged inequality with the exponential function, we can derive the series of inequalities
INEQUALITIES ON THE LINE SEGMENT

\[
H(a, b) \leq \frac{H(a, c)H(c, d)H(d, b)}{32}
\leq \frac{I^{-1}(a^{-1}, c^{-1})I^{-1}(c^{-1}, d^{-1})I^{-1}(d^{-1}, b^{-1})}{32}
\leq (ac^3d^3b)^{1/8} \leq G(a, b),
\]  

(47)

refining the harmonic-geometric mean inequality.

Applying Equation (37) to the exponential function \( f(x) = e^x \) using substitutions \( a \mapsto \ln a, \ b \mapsto \ln b, \ c \mapsto \ln c, \) and \( d \mapsto \ln d, \) we obtain the series of inequalities

\[
G(a, b) \leq L(c, d) \leq L(a, b)
\leq \frac{1}{2} L(a, c) + \frac{1}{2} L(d, b) \leq A(a, b),
\]  

(48)

refining the geometric-logarithmic-arithmetic mean inequality.

References


