BOUNDENESS OF TOEPLITZ TYPE OPERATOR ASSOCIATED TO SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN-ZYGMUND KERNELS ON $L^p$ SPACES WITH VARIABLE EXponent

LIJUAN TONG and JINSONG PAN
Hunan Mechanical and Electrical Polytechnic
Changsha 410151
P. R. China
e-mail: 445927988@qq.com

Abstract

In this paper, the boundedness for some Toeplitz type operator related to some singular integral operator with variable Calderón-Zygmund kernels on $L^p$ spaces with variable exponent is obtained by using a sharp estimate of the operator.

1. Introduction

As the development of the singular integral operators (see [6, 19]), their commutators have been well studied (see [2, 17, 18]). In [1], some singular integral operators with variable Calderón-Zygmund kernels are introduced, and the boundedness for the operators and their
commutators are obtained (see [11, 12, 13, 15, 20]). In [8, 10, 14], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators are obtained. In the last years, a theory of $L^p$ spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations, and elasticity (see [3, 4, 5, 16] and their references). Karlovich and Lerner study the boundedness of the commutators of singular integral operators on $L^p$ spaces with variable exponent (see [7]). Motivated by these papers, the main purpose of this paper is to introduce some Toeplitz type operator related to some singular integral operator with variable Calderón-Zygmund kernels and prove the boundedness for the operator on $L^p$ spaces with variable exponent by using a sharp estimate of the operator.

2. Preliminaries and Results

First, let us introduce some notations. Throughout this paper, $Q$ will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function $f$ and $\delta > 0$, the sharp function of $f$ is defined by

$$f_\delta^\#(x) = \sup_{Q \in \mathcal{E}} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^{\delta} \, dy \right)^{1/\delta},$$

where, and in what follows, $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$. It is well-known that (see [6, 19])

$$f_\delta^\#(x) \approx \inf_{Q \in \mathcal{E}} \left( \frac{1}{|Q|} \int_Q |f(y) - c|^{\delta} \, dy \right)^{1/\delta}.$$

We write $f^\# = f_1^\#$ if $\delta = 1$. We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. Let $M$ be the Hardy-Littlewood maximal operator defined by
\[ M(f)(x) = \sup_{Q \in x} |Q|^{-1} \int_Q |f(y)| \, dy. \]

For \( k \in \mathbb{N} \), we denote by \( M^k \) the operator \( M \) iterated \( k \) times, i.e., \( M^1(f)(x) = M(f)(x) \) and

\[ M^k(f)(x) = M(M^{k-1}(f))(x) \quad \text{when} \quad k \geq 2. \]

Let \( \Phi \) be a Young function and \( \widetilde{\Phi} \) be the complementary associated to \( \Phi \), we denote that the \( \Phi \)-average by, for a function \( f \),

\[ \|f\|_{\Phi, Q} = \inf \left\{ \lambda \geq 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}, \]

and the maximal function associated to \( \Phi \) by

\[ M_{\Phi}(f)(x) = \sup_{Q \in x} \|f\|_{\Phi, Q}. \]

The Young functions to be using in this paper are \( \Phi(t) = t(1 + \log t)^r \) and \( \widetilde{\Phi}(t) = \exp(t^{1/r}) \), the corresponding average and maximal functions denoted by \( \| \cdot \|_{L(\log L)^r, Q}, M_{L(\log L)^r} \) and \( \| \cdot \|_{\exp L^{1/r}, Q}, M_{\exp L^{1/r}} \).

Following [17, 18], we know the generalized Hölder’s inequality:

\[ \frac{1}{|Q|} \int_Q |f(y)g(y)| \, dy \leq \|f\|_{\Phi, Q} \|g\|_{\widetilde{\Phi}, Q}, \]

and the following inequality, for \( r, r_j \geq 1, j = 1, \ldots, l \) with \( 1/r = 1/r_1 + \cdots + 1/r_l \), and any \( x \in \mathbb{R}^n, b \in BMO(\mathbb{R}^n) \),

\[ \|f\|_{L(\log L)^{1/r}, Q}^r \leq M_{L(\log L)^{1/r}}(f) \leq CM_{L(\log L)^{1/r}}(f) \leq CM^{1+1}(f), \]

\[ \|f - f_Q\|_{\exp L^r, Q} \leq C\|f\|_{BMO}, \]

\[ |f_{2^{k+1}Q} - f_{2Q}| \leq C\|f\|_{BMO}. \]
The non-increasing rearrangement of a measurable function \( f \) on \( \mathbb{R}^n \) is defined by
\[
f^*(t) = \inf \{ \lambda > 0 : \| x \in \mathbb{R}^n : |f(x)| > \lambda \| \leq t \} \quad (0 < t < \infty).
\]

For \( \lambda \in (0, 1) \) and a measurable function \( f \) on \( \mathbb{R}^n \), the local sharp maximal function of \( f \) is defined by
\[
M^\#_\lambda f(x) = \sup_{Q \subset C} \inf_{c \in C} (f - c)c_Q^*(\lambda Q).
\]

Let \( p : \mathbb{R}^n \rightarrow [1, \infty) \) be a measurable function. Denote by \( L^{p(\cdot)}(\mathbb{R}^n) \) the sets of all Lebesgue measurable functions \( f \) on \( \mathbb{R}^n \) such that \( m(\lambda f, p) < \infty \) for some \( \lambda = \lambda(f) > 0 \), where
\[
m(f, p) = \int_{\mathbb{R}^n} |f(x)|^p(x) \, dx.
\]

The sets become a Banach spaces with respect to the following norm:
\[
\| f \|_{L^{p(\cdot)}} = \inf \{ \lambda > 0 : m(\lambda f, p) \leq 1 \}.
\]

Denote by \( M(\mathbb{R}^n) \) the sets of all measurable functions \( p : \mathbb{R}^n \rightarrow [1, \infty) \) such that the Hardy-Littlewood maximal operator \( M \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \) and the following holds:
\[
1 < p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x), \quad \text{ess sup}_{x \in \mathbb{R}^n} p(x) = p_+ < \infty.
\]

In recent years, the boundedness of classical operators on spaces \( L^{p(\cdot)}(\mathbb{R}^n) \) have attracted a great attention (see [3, 4, 5, 16] and their references).

In this paper, we will study some singular integral operator as following (see [1]):
Definition 1. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. $K$ is said to be a Calderón-Zygmund kernels, if

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) $\Omega$ is homogeneous of degree zero;

(c) $\int \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of $R^n$.

Definition 2. Let $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. $K$ is said to be a variable Calderón-Zygmund kernels, if

(d) $(K(x, \cdot))$ is a Calderón-Zygmund kernels for a.e. $x \in R^n$;

(e) $\max_{|1| \leq 2^n} \left\| \Omega(x, y) \right\|_{L^x(R^n \times \Sigma)} = L < \infty$.

Moreover, let $b$ be a locally integrable function on $R^n$ and $T$ be the singular integral operator with variable Calderón-Zygmund kernels as

$$T(f)(x) = \int_{R^n} K(x, x-y)f(y)dy,$$

where $K(x, x-y) = \frac{\Omega(x, x-y)}{|x-y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernels.

Let $b$ be a locally integrable function on $R^n$ and $T$ be the singular integral operator with variable Calderón-Zygmund kernels. The Toeplitz type operator associated to $T$ are defined by

$$T_b = \sum_{k=1}^m T^{k,1} M_b T^{k,2},$$
where $T^{k,1}$ are the singular integral operator $T$ with variable Calderón-Zygmund kernels or $\pm I$ (the identity operator), $T^{k,2}$ are the linear operators for $k = 1, \ldots, m$ and $M_b(f) = bf$.

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator $T_b$. The Toeplitz type operators are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [18]). In [1, 15], the boundedness of the singular integral operator with variable Calderón-Zygmund kernels and their commutator are obtained. Our works are motivated by these papers. The main purpose of this paper has twofold, first, we establish a sharp estimate for the operator $T_b$, and second, we prove the boundedness for the operator on $L^p$ spaces with variable exponent by using the sharp estimate.

We shall prove the following theorems:

**Theorem 1.** Let $T$ be the singular integral operators with variable Calderón-Zygmund kernel as Definition 2, $0 < \delta < 1$ and $b \in \text{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that for any $f \in L^\infty_0(R^n)$ and $\tilde{x} \in R^n$,

$$(T_b(f))_{b}^\#(\tilde{x}) \leq C\|g\|_{\text{BMO}} \sum_{k=1}^{m} M^2(T^{k,2}(f))(\tilde{x}).$$

**Theorem 2.** Let $T$ be the singular integral operators with variable Calderón-Zygmund kernels as Definition 2, $p(\cdot) \in M(R^n)$ and $b \in \text{BMO}(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^{p(\cdot)}(R^n)$ for $k = 1, \ldots, m$, then $T_b$ is bounded on $L^{p(\cdot)}(R^n)$, that is,

$$\|T_b(f)\|_{L^{p(\cdot)}} \leq C\|g\|_{\text{BMO}} \|f\|_{L^{p(\cdot)}}.$$
Corollary. Let \([b, T](f) = b T(f) - T(bf)\) be the commutator generated by the singular integral operator \(T\) with variable Calderón-Zygmund kernels and \(b\). Then Theorems 1 and 2 hold for \([b, T]\).

3. Proof of Theorems

To prove the theorems, we need the following lemmas:

Lemma 1 ([6, p.485]). Let \(0 < p < q < \infty\). We define that, for any function \(f \geq 0\) and \(1/r = 1/p - 1/q\),

\[
\|f\|_{WL^q} = \sup_{\lambda > 0} \|\{x \in \mathbb{R}^n : f(x) > \lambda\}\|^{1/q}, \quad N_{p,q}(f) = \sup_E \|f_{\chi E}\|_{L^p} / \|\chi_E\|_{L'}
\]

where the sup is taken for all measurable sets \(E\) with \(0 < |E| < \infty\). Then

\[
\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q / (q - p))^{1/p} \|f\|_{WL^q}.
\]

Lemma 2 ([18]). Let \(r_j \geq 1\) for \(j = 1, \ldots, l\), we denote that \(1/r = 1/\eta_1 + \cdots + 1/\eta_l\). Then

\[
\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x) g(x)| \, dx \leq \|f\|_{\exp L^n, Q} \cdots \|f\|_{\exp L^n, Q} \|g\|_{L(\log L)^{1/r}, Q}.
\]

Lemma 3 ([1]). Let \(T\) be the singular integral operators with variable Calderón-Zygmund kernels as Definition 2. Then \(T\) is bounded from \(L^1(\mathbb{R}^n)\) to \(WL^1(\mathbb{R}^n)\).

Lemma 4 ([16]). Let \(p : \mathbb{R}^n \to [1, \infty)\) be a measurable function satisfying (1). Then \(L^p_0(\mathbb{R}^n)\) is dense in \(L^p(\mathbb{R}^n)\).

Lemma 5 ([7, 9]). Let \(\delta > 0\), \(0 < \lambda < 1\), and \(f \in L^\delta_{\text{loc}}(\mathbb{R}^n)\). Then

\[
M^\#_{\lambda}(f)(x) \leq (1 / \lambda)^{1/\delta} f^\#_{\delta}(x).
\]
Lemma 6 ([16]). Let \( f \in L^1_{\text{loc}}(R^n) \) and \( g \) be a measurable function satisfying
\[
\left\{ x \in R^n : |g(x)| > \alpha \right\} < \infty \text{ for all } \alpha > 0.
\]
Then
\[
\int_{R^n} |f(x)g(x)|dx \leq C_n \int_{R^n} M_{b_n}^\#(f)(x)M(g)(x)dx.
\]

Lemma 7 ([9]). Let \( p : R^n \to [1, \infty) \) be a measurable function satisfying (1). If \( f \in L^p(R^n) \) and \( g \in L^{p'}(R^n) \) with \( p'(x) = p(x)/(p(x) - 1) \). Then \( fg \) is integrable on \( R^n \) and
\[
\int_{R^n} |f(x)g(x)|dx \leq C\|f\|_{L^p}\|g\|_{L^{p'}}.
\]

Lemma 8 ([9]). Let \( p : R^n \to [1, \infty) \) be a measurable function satisfying (1). Set
\[
\|f\|_{L^p} = \sup \left\{ \int_{R^n} |f(x)g(x)|dx : f \in L^p(R^n), g \in L^{p'}(R^n) \right\}.
\]
Then \( \|f\|_{L^p} \leq \|f\|_{L^p'} \leq C\|f\|_{L^p} \).

Proof of Theorem 1. It suffices to prove for \( f \in L^\infty_0(R^n) \) and some constant \( C_0 \), the following inequality holds:
\[
\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0|^\delta dx \right)^{1/\delta} \leq C\|b\|_{\text{BMO}} \sum_{k=1}^m M^2(T^{k,2}(f))(\bar{x}).
\]
Without loss of generality, we may assume \( T^{k,1} \), are \( T(k = 1, \ldots, m) \). Fix a cube \( Q = Q(x_0, d) \) and \( \bar{x} \in Q \). We write, by \( T_1(g) = 0 \),
\[
T_b(f)(x) = T_{b - b_2Q}(f)(x)
\]
\[
= T_{(b - b_2Q)_{2Q}}(f)(x) + T_{(b - b_2Q)_{2Q}'}(f)(x)
\]
\[
= f_1(x) + f_2(x).
\]
Then
\[
\left( \frac{1}{|Q|} \int_Q \left| T_b(f)(x) - f_2(x_0) \right|^\delta \, dx \right)^{1/\delta} \leq C \left( \frac{1}{|Q|} \int_Q |f_1(x)|^\delta \, dx \right)^{1/\delta} + C \left( \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)|^\delta \, dx \right)^{1/\delta} = I + II.
\]

For \( I \), by Lemmas 1, 2, and 3, we obtain
\[
\left( \frac{1}{|Q|} \int_Q \left| T^{k,1} M_{(b-b_2Q)} T^{k,2}(f)(x) \right|^\delta \, dx \right)^{1/\delta} \leq |Q|^{-1} \left\| T^{k,1} M_{(b-b_2Q)} T^{k,2}(f) \right\|_{L^\delta} \leq C |Q|^{-1} \left\| T^{k,1} M_{(b-b_2Q)} T^{k,2}(f) \right\|_{L^1} \leq C |Q|^{-1} \int_{2Q} |b(x) - b_{2Q}| \left| T^{k,2}(f)(x) \right| \, dx \leq C \| b - b_{2Q} \|_{\exp L, 2Q} \left\| T^{k,2}(f) \right\|_{L(\log L), 2Q} \leq C \| b \|_{BMO} M^2(T^{k,2}(f))(\bar{x}),
\]

thus,
\[
I \leq \sum_{k=1}^m \left( \frac{C}{|Q|} \int_Q \left| T^{k,1} M_{(b-b_2Q)} T^{k,2}(f)(x) \right|^\delta \, dx \right)^{1/\delta} \leq C \| b \|_{BMO} \sum_{k=1}^m M^2(T^{k,2}(f))(\bar{x}).
\]
For II, by [1], we know that

\[ T(f)(x) = \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} a_{uv}(x) \int_{\mathbb{R}^n} \frac{Y_{uv}(x-y)}{|x-y|^{n+m}} f(y) dy, \]

where \( g_u \leq Cu^{n-2}, \|a_{uv}\|_{L^\infty} \leq Cu^{-2n}, |Y_{uv}(x-y)| \leq Cu^{n/2-1}, \) and

\[ \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| \leq Cu^{n/2} \frac{|x-x_0|}{|x_0-y|^{n+1}}, \]

for \( |x-y| > 2|x_0-x| > 0. \) Then, we get, for \( x \in Q, \)

\[ |T^{h,1}M_{b-b2Q}(2Q)(f)(x) - T^{h,1}M_{b-b2Q}(2Q)(M_{h,2}(f)(x_0))| \]

\[ \leq \int_{(2Q)^c} |b(y) - b_{2Q}| \left| K(x, x-y) - K(x_0, x_0-y) \right| |T^{h,2}(f)(y)| dy \]

\[ = \sum_{j=1}^{\infty} \int_{2^j d \leq |x-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| \left| K(x, x-y) - K(x_0, x_0-y) \right| |T^{h,2}(f)(y)| dy \]

\[ \leq C \sum_{j=1}^{\infty} \int_{2^j d \leq |x-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} |a_{uv}(x)| \]

\[ \times \left| \frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n} \right| |T^{h,2}(f)(y)| dy \]

\[ \leq C \sum_{j=1}^{\infty} \int_{2^j d \leq |x-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| |x-x_0| \frac{|x_0-y|^{n+1}}{|x_0-y|^{n+1}} |T^{h,2}(f)(y)| dy \]

\[ \leq C \sum_{j=1}^{\infty} \frac{d}{(2^{j+1} d)^{n+1}} \int_{2^j d Q} |b(y) - b_{2Q}| |T^{h,2}(f)(y)| dy \]

\[ \leq C \sum_{j=1}^{\infty} \frac{2^{-j}}{|2^{j+1} Q|} \int_{2^j d Q} |b(y) - b_{2Q}| |T^{h,2}(f)(y)| dy \]
\[ \leq C \sum_{j=1}^{\infty} 2^{-j} \|b - b_{2jQ}\|_{L, 2^{j+1}Q} \|T^{h, 2}(f)\|_{L(\log L), 2^{j+1}Q} \]

\[ \leq C \sum_{j=1}^{\infty} j2^{-j} \|\|_{BMO} M^2(T^{h, 2}(f))(\bar{x}) \]

\[ \leq C \|\|_{BMO} M^2(T^{h, 2}(f))(\bar{x}), \]

thus,

\[ II \leq C \frac{1}{|Q|} \sum_{Q=1}^{m} \int_{Q} T^{h, 1}(b_{-b_{2jQ}}) T^{h, 2}(f)(x) - T^{h, 1}(b_{-b_{2jQ}}) T^{h, 2}(f)(x_0) \|dx \]

\[ \leq C \|\|_{BMO} \sum_{Q=1}^{m} M^2(T^{h, 2}(f))(\bar{x}). \]

This completes the proof of Theorem 1.

**Proof of Theorem 2.** By Lemmas 4-7, we get, for \( f \in L_{0}^{\infty}(R^n) \) and \( g \in L^{p(\cdot)}(R^n) \),

\[ \int_{R^n} |T_b(f)(x)g(x)|dx \leq C \int_{R^n} M_{T_b}^\#(T_b(f))(x)M(g)(x)dx \]

\[ \leq C \int_{R^n} (T_b(f))^\#(x)M(g)(x)dx \]

\[ \leq C \|\|_{BMO} \sum_{k=1}^{m} \int_{R^n} M^2(T^{h, 2}(f))(x)M(g)(x)dx \]

\[ \leq C \|\|_{BMO} \sum_{k=1}^{m} \|M^2(T^{h, 2}(f))\|_{L^{p(\cdot)}} \|M(g)\|_{L^{p(\cdot)}} \]

\[ \leq C \|\|_{BMO} \sum_{k=1}^{m} \|T^{h, 2}(f)\|_{L^{p(\cdot)}} \|M(g)\|_{L^{p(\cdot)}} \]

\[ \leq C \|\|_{BMO} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p(\cdot)}}, \]
thus, by Lemma 8,
\[ \|T_b(f)\|_{L^p} \leq \|\mathbb{BMO}\|_{L^p}. \]

This completes the proof of Theorem 2.

References


